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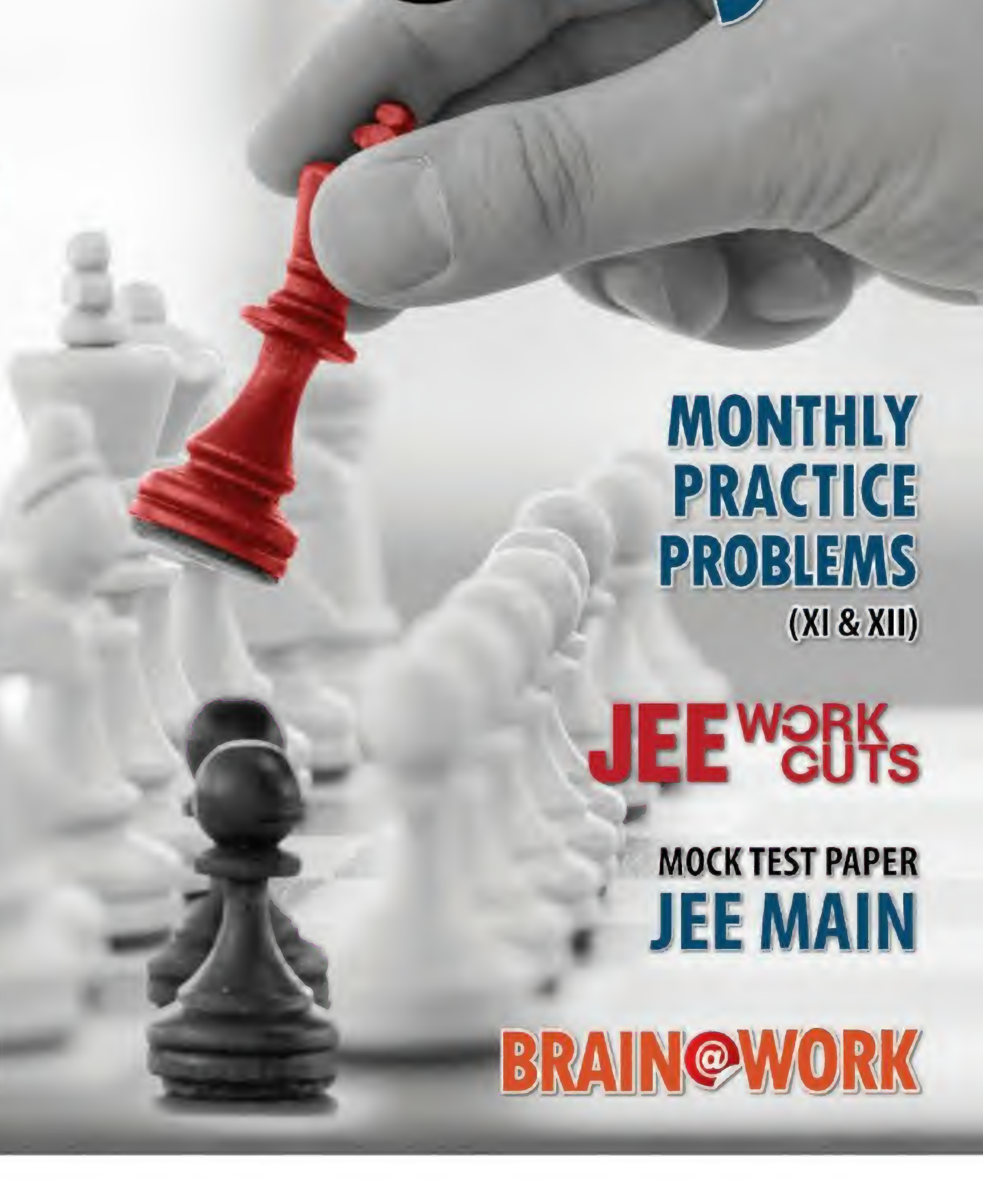

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Vol. XXXVI No. 1 January 2018

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406, Taj Apartment, Near Safdarjung Hospital,
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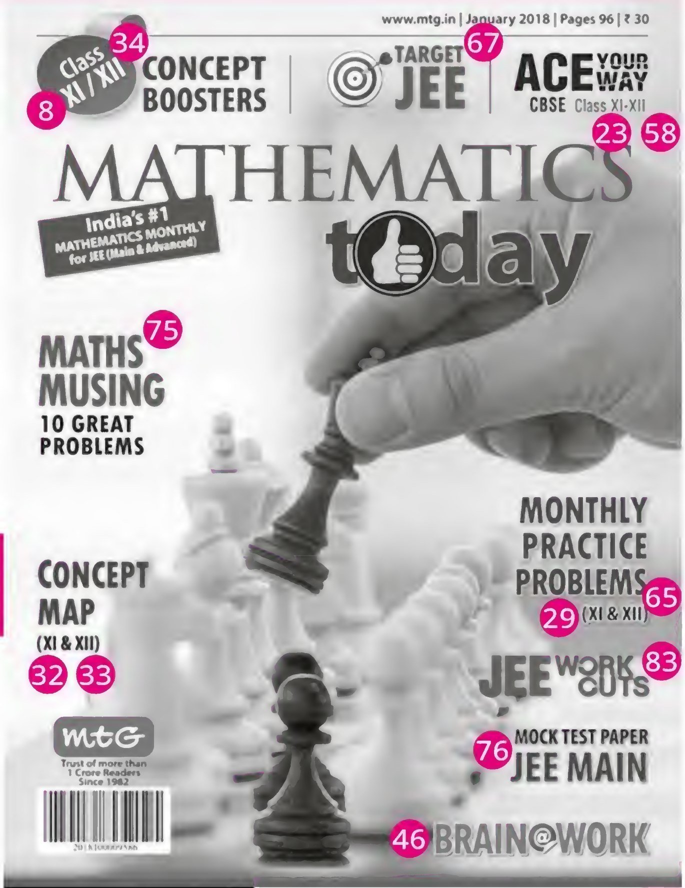

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2018

Printed and Published by Mahabir Singh on behalf of MTG Learning Media Pvt. Ltd. Printed at HT Media Ltd., B-2, Sector-63, Noida, UP-201307 and published at 406, Taj Apartment, Ring Road, Near Safdarjung Hospital, New Delhi - 110029.
Editor : Anil Ahlawat

CONCEPT BOOSTERS

Ellipse and Hyperbola

Class XI

This column is aimed at Class XI students so that they can prepare for competitive exams such as JEE Main/Advanced, etc. and be also in command of what is being covered in their school as part of NCERT syllabus. The problems here are a happy blend of the straight and the twisted, the simple and the difficult and the easy and the challenging.

*ALOK KUMAR, B.Tech, IIT Kanpur

ELLIPSE

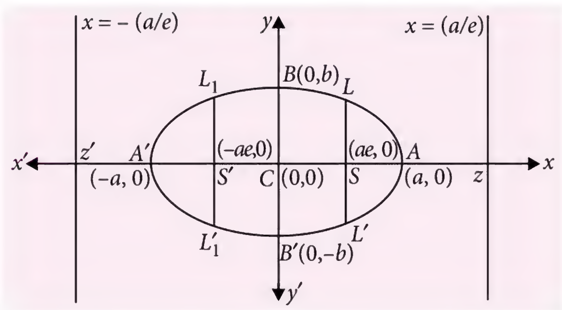
- The Ellipse is a conic whose eccentricity is less than unity *i.e.*, $e < 1$.
- Standard equation of an ellipse referred to its principal axes along the co-ordinate axes is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where $a > b$

  Eccentricity : $e=\sqrt{1-\frac{b^2}{a^2}} \Rightarrow a^2e^2 = a^2 - b^2 (0 < e < 1)$

  Foci : $S \equiv (ae, 0)$ and $S' \equiv (-ae, 0)$

  The equation of directrices are $x=\frac{a}{e}$ and $x=-\frac{a}{e}$

  Vertices : $A' \equiv (-a, 0)$ and $A \equiv (a, 0)$



- The line segment $A'A$ of length $2a(a > b)$ is called the major axis of the ellipse.
- The y-axis intersects the ellipse at the points $B' \equiv (0, -b)$ and $B \equiv (0, b)$. The line segment $B'B$ of length $2b(b < a)$ is called the minor axis of the ellipse.
- The major and minor axis together are called principal axis of the ellipse.
- A chord which passes through focus is called a focal chord and which passes through the centre is called a diameter of the conic.
- A chord perpendicular to the major axis is called a double ordinate.
- The focal chord perpendicular to the major axis is called the latus rectum. Length of latus rectum (LL') $=\frac{2b^2}{a}=2a(1-e^2)$
- The sum of the focal distances of any point on the ellipse is equal to the major axis. Hence, distance of focus from the extremity of a minor axis is equal to semi major axis.
  i.e., $BS = CA$.
- The point $P(x_1, y_1)$ lies outside, inside or on the ellipse according as, $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1 >, <$ or $=0$ (respectively)
- A circle described on major axis as diameter is called the auxiliary circle. Let Q be a point on the auxiliary circle $x^2 + y^2 = a^2$ such that QP produced is perpendicular to the x-axis then P and Q are called as the corresponding points on the ellipse and the auxiliary circle respectively 'θ' is called the eccentric angle of the point P on the ellipse $(0 \le \theta < 2\pi)$.

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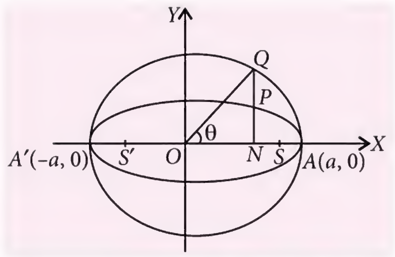
He trains IIT and Olympiad aspirants.



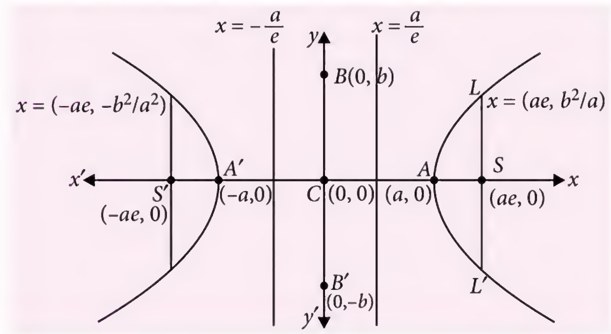
The equations $x = a\cos\theta$ and $y = b\sin\theta$ together represent the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where θ is a parameter.

Note : If $P(\theta) \equiv (a\cos\theta, b\sin\theta)$ is on the ellipse then, $Q(\theta) \equiv (a\cos\theta, a\sin\theta)$ is on the auxiliary circle.

- The line $y = mx + c$ meets the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in two points real, coincident or imaginary according as c^2 is <, = or > $a^2m^2 + b^2$.
  Hence, $y = mx + c$ is tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ if $c^2 = a^2m^2 + b^2$.
  The equation to the chord of the ellipse joining two points with eccentric angles α and β is given by $\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right)+\frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right)=\cos\left(\frac{\alpha-\beta}{2}\right)$.
- $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$ is tangent to the ellipse at (x_1, y_1).
  For general ellipse replace x^2 by xx_1, y^2 by yy_1, $2x$ by $x + x_1$, $2y$ by $y + y_1$, $2xy$ by $xy_1 + yx_1$.
- $y = mx \pm \sqrt{a^2m^2+b^2}$ is tangent to the ellipse for all values of m.
- $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$ is tangent to the ellipse at the point $(a\cos\theta, b\sin\theta)$.
- The eccentric angles of point of contact of two parallel tangents differ by an angle π.
- Equation of the normal at (x_1, y_1) is
  $$\frac{a^2x}{x_1}-\frac{b^2y}{y_1} = a^2 - b^2 = a^2e^2.$$
- Equation of the normal at the point $(a\cos\theta, b\sin\theta)$ is $ax\cdot\sec\theta - by\cdot\operatorname{cosec}\theta = a^2 - b^2$.
- Equation of a normal in terms of its slope 'm' is y
  $$y = mx - \frac{(a^2-b^2)m}{\sqrt{a^2+b^2m^2}}.$$
- Locus of the point of intersection of the tangents which meet at right angles is called the director circle. The equation to this locus is $x^2 + y^2 = a^2 + b^2$ *i.e.* a circle whose centre is the centre of the ellipse and whose radius is the length of the line joining the ends of the major and minor axis.

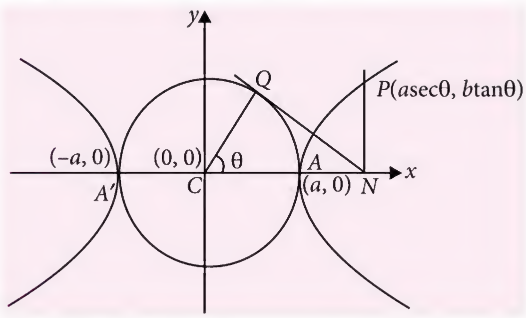
HYPERBOLA

- The hyperbola is a conic whose eccentricity is greater than unity i.e., $e > 1$.
- Standard equation of the hyperbola is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.
  Eccentricity : $e = \sqrt{1+\frac{b^2}{a^2}} \Rightarrow a^2e^2 = a^2 + b^2$
  Foci : $S \equiv (ae , 0)$ and $S' \equiv (-ae , 0)$.

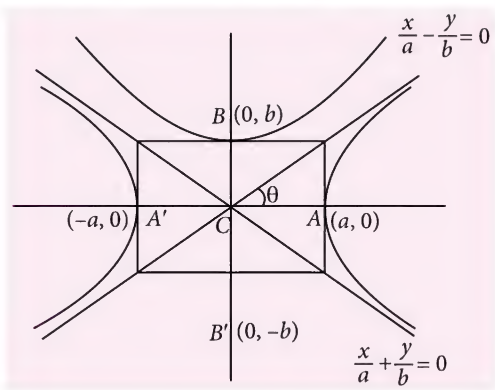


  Equation of directrices : $x = \frac{a}{e}$ and $x = -\frac{a}{e}$
  Vertices : $A \equiv (a, 0)$ and $A' \equiv (-a, 0)$.
  Latus rectum : $l = \frac{2b^2}{a} = 2a(e^2-1)$.
- The line segment $A'A$ of length $2a$ in which the foci S' and S both lie is called the transverse axis of the hyperbola.
- The line segment $B'B$ between the two points $B' \equiv (0, -b)$ and $B \equiv (0, b)$ is called as the conjugate axis of the hyperbola. The transverse axis and the conjugate axis of the hyperbola are together called the principal axes of the hyperbola.
- The difference of the focal distances of any point P on the hyperbola is constant and equal to transverse axis *i.e.* $||PS| - |PS'|| = 2a$. The distance SS' = focal length.
- Two hyperbolas such that transverse & conjugate axes of one hyperbola are respectively the conjugate & the transverse axes of the other are called conjugate hyperbolas of each other .
  i.e. $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and $-\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are conjugate hyperbola of each other.

- If e_1 and e_2 are the eccentricities of the hyperbola and its conjugate then $e_1^{-2} + e_2^{-2} = 1$.
- The particular kind of hyperbola in which the lengths of the transverse and conjugate axis are equal is called an equilateral hyperbola.
  Note : Eccentricity of the rectangular hyperbola is $\sqrt{2}$ and the length of its latus rectum is equal to its transverse or conjugate axis.



- A circle drawn with centre C & transverse axis as a diameter is called the auxiliary circle of the hyperbola. Equation of the auxiliary circle is $x^2 + y^2 = a^2$. Note from the figure that P and Q are called the "corresponding points" on the hyperbola and the auxiliary circle. 'θ' is called the eccentric angle of the point 'P' on the hyperbola ($0 \le \theta < 2\pi$).
- The point (x_1, y_1) will be outside, on or inside the hyperbola according as $\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1 >, =$ or < 0 (respectively)
- The straight line $y = mx + c$ is a secant, a tangent or passes outside the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ according as $c^2 >, =, < a^2 m^2 - b^2$.
  Hence, $y = mx + c$ is tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ if $c^2 = a^2m^2 - b^2$.
- Equation of the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point (x_1, y_1) is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$.
  In general two tangents can be drawn from an external point (x_1, y_1) to the hyperbola and they are $y - y_1 = m_1 (x - x_1)$ and $y - y_2 = m_2 (x - x_2)$, where m_1 and m_2 are roots of the equation $(x_1^2 - a^2)m^2 - 2x_1y_1m + y_1^2 + b^2 = 0$.
- Equation of the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $(a \sec\theta, b \tan\theta)$ is $\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$.
  Also, $y = mx \pm \sqrt{a^2m^2 - b^2}$ can be taken as the tangents to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.
- Equation of a chord joining α and β is $\frac{x}{a}\cos\frac{\alpha-\beta}{2} - \frac{y}{b}\sin\frac{\alpha+\beta}{2} = \cos\frac{\alpha+\beta}{2}$.
- The equation of the normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $p(x_1, y_1)$ on it is
  $$\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2 = a^2e^2.$$
- The equation of the normal at the point $P(a \sec\theta, b \tan\theta)$ on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is
  $$\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2 = a^2e^2.$$
- The locus of the intersection of tangents which are at right angles is known as the director circle of the hyperbola. The equation to the director circle is $x^2 + y^2 = a^2 - b^2$.
  If $b^2 < a^2$ this circle is real; if $b^2 = a^2$ the radius of the circle is zero and it reduces to a point circle at the origin. In this case, the centre is the only point from which the tangents at right angles can be drawn to the curve. If $b^2 > a^2$, the radius of the circle is imaginary, so that there is no such circle and so no tangents at right angle can be drawn to the curve.
- Let $y = mx + c$ is the asymptote of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.
  Solving these two we get the quadratic as
  $(b^2 - a^2m^2)x^2 - 2a^2mcx - a^2(b^2 + c^2) = 0$ ...(i)



In order that $y = mx + c$ be an asymptote, both roots of equation (i) must approach infinity, the conditions for which are : coeff. of $x^2 = 0$ and coeff. of $x = 0$.

$\Rightarrow \quad b^2 - a^2m^2 = 0 \quad$ or $\quad m = \pm\frac{b}{a}$

and $a^2mc = 0 \quad \Rightarrow \quad c = 0$.

$\therefore$ Equations of asymptote are

$$\frac{x}{a}+\frac{y}{b}=0 \text{ and } \frac{x}{a}-\frac{y}{b}=0$$

On combining equation to the asymptotes, we get

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$$

- When $b = a$, then, the asymptotes of the rectangular hyperbola $x^2 - y^2 = a^2$ or $y = \pm x$ which are at right angles.
- If a hyperbola is equilateral then the conjugate hyperbola is also equilateral.
- A hyperbola and its conjugate have the same asymptote.
- The equation of the pair of asymptotes differ the hyperbola and the conjugate hyperbola by the same constant only.
- Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$. Then the point of intersection of $\frac{\partial f}{\partial x}=0$ and $\frac{\partial f}{\partial y}=0$ gives the centre of the hyperbola.

Important Points on Rectangular Hyperbola

- Equation is $xy = c^2$ with parametric representation $x = ct,\ y = c/t,\ t \in R$.
- Equation of a chord joining the points $P(t_1)$ and $Q(t_2)$ is $x + t_1t_2\, y = c(t_1 + t_2)$.
- Equation of the tangent at $P(x_1, y_1)$ is $xy_1 + x_1y = 2c^2$ and at $P(ct, c/t)$ is $\frac{x}{t}+ty=2c$.
- Chord with a given middle point as (h, k) is $kx + hy = 2hk$.
- Equation of the normal at $P(ct, c/t)$ is $xt^3 - yt = c(t^4 - 1)$.
- Vertex of this hyperbola is (c, c) and $(-c, -c)$, focus is $(\sqrt{2}c, \sqrt{2}c)$ and $(-\sqrt{2}c, -\sqrt{2}c)$, the directrices are $x + y = \pm\sqrt{2}\,c$ and latus rectum $= 2\sqrt{2}c$.

PROBLEMS

Single Correct Answer Type

1. The eccentricity of the hyperbola conjugate to $x^2 - 3y^2 = 2x + 8$ is
(a) $\frac{2}{\sqrt{3}}$ (b) $\sqrt{3}$
(c) 2 (d) none of these

2. The pole of the straight line $x + 4y = 4$ with respect to ellipse $x^2 + 4y^2 = 4$ is
(a) (1, 4) (b) (1, 1) (c) (4, 1) (d) (4, 4)

3. In the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, the equation of diameter conjugate to the diameter $y=\frac{b}{a}x$, is
(a) $y=-\frac{b}{a}x$ (b) $y=-\frac{a}{b}x$
(c) $x=-\frac{b}{a}y$ (d) none of these

4. Minimum area of the triangle by any tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with the coordinate axes is
(a) $\frac{a^2+b^2}{2}$ (b) $\frac{(a+b)^2}{2}$
(c) ab (d) $\frac{(a-b)^2}{2}$

5. The eccentricity of the ellipse $25x^2 + 16y^2 - 150x - 175 = 0$ is
(a) 2/5 (b) 2/3
(c) 4/5 (d) 3/5

6. If $5x^2 + \lambda y^2 = 20$ represents a rectangular hyperbola, then λ equals
(a) 5 (b) 4
(c) –5 (d) none of these

7. The locus of the point of intersection of the lines $bxt - ayt = ab$ and $bx + ay = abt$ is
(a) a parabola (b) an ellipse
(c) a hyperbola (d) none of these

8. The eccentricity of the hyperbola can never be equal to
(a) $\sqrt{\frac{9}{5}}$ (b) $2\sqrt{\frac{1}{9}}$ (c) $3\sqrt{\frac{1}{8}}$ (d) 2

9. The locus of the centre of a circle, which touches externally the given two circles, is
(a) circle (b) parabola
(c) hyperbola (d) ellipse

10. The foci of the hyperbola $9x^2 - 16y^2 = 144$ are
(a) $(\pm 4, 0)$ (b) $(0, \pm 4)$
(c) $(\pm 5, 0)$ (d) $(0, \pm 5)$

11. The equation $x^2 + 4xy + y^2 + 2x + 4y + 2 = 0$ represents
(a) an ellipse (b) a pair of straight lines
(c) a hyperbola (d) none of these

12. The auxiliary equation of circle of hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, is
(a) $x^2 + y^2 = a^2$ (b) $x^2 + y^2 = b^2$
(c) $x^2 + y^2 = a^2 + b^2$ (d) $x^2 + y^2 = a^2 - b^2$

13. If the line $y = 2x + \lambda$ be a tangent to the hyperbola $36x^2 - 25y^2 = 3600$, then $\lambda =$
(a) 16 (b) −16
(c) ±16 (d) none of these

14. The equation of the tangents to the conic $3x^2 - y^2 = 3$ perpendicular to the line $x + 3y = 2$ is
(a) $y=3x\pm\sqrt{6}$ (b) $y=6x\pm\sqrt{3}$
(c) $y=x\pm\sqrt{6}$ (d) $y = 3x \pm 6$

15. The locus of the point of intersection of any two perpendicular tangents to the hyperbola is a circle which is called the director circle of the hyperbola, then the equation of this circle is
(a) $x^2 + y^2 = a^2 + b^2$ (b) $x^2 + y^2 = a^2 - b^2$
(c) $x^2 + y^2 = 2ab$ (d) none of these

16. The equation of the tangents to the hyperbola $3x^2 - 4y^2 = 12$ which cuts equal intercepts from the axes, are
(a) $y + x = \pm1$ (b) $y - x = \pm1$
(c) $3x + 4y = \pm1$ (d) $3x - 4y = \pm1$

17. If m_1 and m_2 are the slopes of the tangents to the hyperbola $\frac{x^2}{25}-\frac{y^2}{16}=1$ which pass through the point (6, 2), then
(a) $m_1+m_2=\frac{24}{11}$ (b) $m_1m_2=\frac{2}{11}$
(c) $m_1+m_2=\frac{48}{11}$ (d) $m_1m_2=\frac{11}{20}$

18. The value of m for which $y = mx + 6$ is a tangent to the hyperbola $\frac{x^2}{100}-\frac{y^2}{49}=1$, is
(a) $\sqrt{\frac{17}{20}}$ (b) $\sqrt{\frac{20}{17}}$
(c) $\sqrt{\frac{3}{20}}$ (d) $\sqrt{\frac{20}{3}}$

19. The equation of the tangent to the conic $x^2 - y^2 - 8x + 2y + 11 = 0$ at (2, 1) is
(a) $x + 2 = 0$ (b) $2x + 1 = 0$
(c) $x - 2 = 0$ (d) $x + y + 1 = 0$

20. If the straight line $x \cos\alpha + y \sin\alpha = p$ be a tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, then
(a) $a^2 \cos^2\alpha + b^2 \sin^2\alpha = p^2$
(b) $a^2 \cos^2\alpha - b^2 \sin^2\alpha = p^2$
(c) $a^2 \sin^2\alpha + b^2 \cos^2\alpha = p^2$
(d) $a^2 \sin^2\alpha - b^2 \cos^2\alpha = p^2$

21. If the tangent on the point $(2 \sec\phi, 3\tan\phi)$ of the hyperbola $\frac{x^2}{4}-\frac{y^2}{9}=1$ is parallel to $3x - y + 4 = 0$, then the value of ϕ is
(a) 45° (b) 60° (c) 30° (d) 75°

22. What is the slope of the tangent line drawn to the hyperbola $xy = a(a \neq 0)$ at the point $(a, 1)$?
(a) $1/a$ (b) $-1/a$ (c) a (d) $-a$

23. The straight line $x+y=\sqrt{2}p$ will touch the hyperbola $4x^2 - 9y^2 = 36$, if
(a) $p^2 = 2$ (b) $p^2 = 5$
(c) $5p^2 = 2$ (d) $2p^2 = 5$

24. Let E be the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ and C be the circle $x^2 + y^2 = 9$. Let P and Q be the points (1, 2) and (2, 1) respectively. Then
(a) Q lies inside C but outside E
(b) Q lies outside both C and E
(c) P lies inside both C and E
(d) P lies inside C but outside E

25. The distance between the directrices of the ellipse $\frac{x^2}{36}+\frac{y^2}{20}=1$ is
(a) 8 (b) 12 (c) 18 (d) 24

26. The equation of the ellipse whose vertices are (±5, 0) and foci are (±4, 0) is
(a) $9x^2 + 25y^2 = 225$ (b) $25x^2 + 9y^2 = 225$
(c) $3x^2 + 4y^2 = 192$ (d) none of these

27. The equation of the ellipse whose one of the vertices is (0, 7) and the corresponding directrix is $y = 12$, is
(a) $95x^2 + 144y^2 = 4655$ (b) $144x^2 + 95y^2 = 4655$
(c) $95x^2 + 144y^2 = 13680$ (d) none of these

28. For the ellipse $3x^2 + 4y^2 = 12$, the length of latus rectum is
(a) $\frac{3}{2}$ (b) 3 (c) $\frac{8}{3}$ (d) $\sqrt{\frac{3}{2}}$

29. Eccentricity of the ellipse $9x^2 + 25y^2 = 225$ is

(a) $\frac{3}{5}$ (b) $\frac{4}{5}$ (c) $\frac{9}{25}$ (d) $\frac{\sqrt{34}}{5}$

30. What is the equation of the ellipse with foci $(\pm 2, 0)$ and eccentricity $= \frac{1}{2}$?

(a) $3x^2 + 4y^2 = 48$ (b) $4x^2 + 3y^2 = 48$
(c) $3x^2 + 4y^2 = 0$ (d) $4x^2 + 3y^2 = 0$

31. The equation of ellipse whose distance between the foci is equal to 8 and distance between the directrices is 18, is

(a) $5x^2 - 9y^2 = 180$ (b) $9x^2 + 5y^2 = 180$
(c) $x^2 + 9y^2 = 180$ (d) $5x^2 + 9y^2 = 180$

32. The centre of the ellipse $4x^2 + 9y^2 - 16x - 54y + 61 = 0$ is

(a) (1,3) (b) (2, 3)
(c) (3, 2) (d) (3, 1)

33. The equation of an ellipse whose eccentricity is 1/2 and the vertices are (4, 0) and (10, 0) is

(a) $3x^2 + 4y^2 - 42x + 120 = 0$
(b) $3x^2 + 4y^2 + 42x + 120 = 0$
(c) $3x^2 + 4y^2 + 42x - 120 = 0$
(d) $3x^2 + 4y^2 - 42x - 120 = 0$

34. The eccentricity of the ellipse $9x^2 + 5y^2 - 18x - 20y - 16 = 0$ is

(a) 1/2 (b) 2/3 (c) 1/3 (d) 3/4

35. The position of the point (1, 3) with respect to the ellipse $4x^2 + 9y^2 - 16x - 54y + 61 = 0$

(a) outside the ellipse (b) on the ellipse
(c) on the major axis (d) on the minor axis

36. The locus of the point of intersection of mutually perpendicular tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, is

(a) a straight line (b) a parabola
(c) a director circle (d) none of these

37. The equation of the tangents drawn at the ends of the major axis of the ellipse $9x^2 + 5y^2 - 30y = 0$, are

(a) $y = \pm 3$ (b) $x = \pm\sqrt{5}$
(c) $y = 0, y = 6$ (d) none of these

38. The line $y = mx + c$ is a normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, if $c =$

(a) $-(2am + bm^2)$ (b) $\frac{(a^2 + b^2)m}{\sqrt{a^2 + b^2m^2}}$

(c) $-\frac{(a^2 - b^2)m}{\sqrt{a^2 + b^2m^2}}$ (d) $\frac{(a^2 - b^2)m}{\sqrt{a^2 + b^2}}$

39. The equation of the normal at the point (2, 3) on the ellipse $9x^2 + 16y^2 = 180$, is

(a) $3y = 8x - 10$ (b) $3y - 8x + 7 = 0$
(c) $8y + 3x + 7 = 0$ (d) $3x + 2y + 7 = 0$

40. If (4, 0) and (–4, 0) be the vertices and (6, 0) and (–6, 0) be the foci of a hyperbola, then its eccentricity is

(a) 5/2 (b) 2 (c) 3/2 (d) $\sqrt{2}$

41. If $(0, \pm 4)$ and $(0, \pm 2)$ be the foci and vertices of a hyperbola, then its equation is

(a) $\frac{x^2}{4} - \frac{y^2}{12} = 1$ (b) $\frac{x^2}{12} - \frac{y^2}{4} = 1$

(c) $\frac{y^2}{4} - \frac{x^2}{12} = 1$ (d) $\frac{y^2}{12} - \frac{x^2}{4} = 1$

42. The eccentricity of the hyperbola $4x^2 - 9y^2 = 16$, is

(a) $\frac{8}{3}$ (b) $\frac{5}{4}$ (c) $\frac{\sqrt{13}}{3}$ (d) $\frac{4}{3}$

43. The eccentricity of the hyperbola $2x^2 - y^2 = 6$ is

(a) $\sqrt{2}$ (b) 2 (c) 3 (d) $\sqrt{3}$

44. The equation $x^2 - 16xy - 11y^2 - 12x + 6y + 21 = 0$ represents

(a) parabola (b) ellipse
(c) hyperbola (d) two straight lines

45. The equation of the hyperbola referred to its axes as axes of coordinate and whose distance between the foci is 16 and eccentricity is $\sqrt{2}$, is

(a) $x^2 - y^2 = 16$ (b) $x^2 - y^2 = 32$
(c) $x^2 - 2y^2 = 16$ (d) $y^2 - x^2 = 16$

46. If the foci of the ellipse $\frac{x^2}{16} + \frac{y^2}{b^2} = 1$ and the hyperbola $\frac{x^2}{144} - \frac{y^2}{81} = \frac{1}{25}$ coincide, then the value of b^2 is

(a) 1 (b) 5 (c) 7 (d) 9

47. The eccentricity of the hyperbola $\frac{\sqrt{1999}}{3}(x^2 - y^2) = 1$ is

(a) $\sqrt{3}$ (b) $\sqrt{2}$ (c) 2 (d) $2\sqrt{2}$

Assertion & Reason Type

Directions : *In the following questions, Statement-1 is followed by Statement-2. Mark the correct choice as :*

(a) Statement-1 is true, Statement-2 is true, Statement-2 is a correct explanation for Statement-1.
(b) Statement-1 is true, Statement-2 is true, Statement-2 is not a correct explanation for Statement-1.
(c) Statement-1 is true, Statement-2 is false.
(d) Statement-1 is false, Statement-2 is true.

48. **Statement-1 :** The foci of the hyperbola $xy = 36$ are $(6\sqrt{2}, 6\sqrt{2})$ and $(-6\sqrt{2}, -6\sqrt{2})$.

Statement-2 : The foci of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are $\left(\pm\sqrt{a^2 - b^2}, 0\right)$.

49. **Statement-1 :** The angle of intersection between the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the circle $x^2 + y^2 = ab$ is $\tan^{-1}\left(\frac{b-a}{\sqrt{ab}}\right)$.

Statement-2 : The point of intersection of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $x^2 + y^2 = ab$ is $\left(\sqrt{\frac{ab}{a+b}}, \sqrt{\frac{ab}{a+b}}\right)$.

50. **Statement-1 :** The condition on a and b for which two distinct chords of the ellipse $\frac{x^2}{2a^2} + \frac{y^2}{2b^2} = 1$ passing through $(a, -b)$ are bisected by $x + y = b$ is $a^2 + 6ab - 7b^2 > 0$.

Statement 2 : Equation of the chord of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ whose mid point is (x_1, y_1) of the form $T = S_1$

i.e. $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1 = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1$.

51. **Statement-1 :** If the point (x, y) lies on the curve $2x^2 + y^2 - 24y + 80 = 0$ then the maximum value of $x^2 + y^2$ is 400.

Statement-2 : The point (x, y) is at a distance of $\sqrt{x^2 + y^2}$ from origin.

Comprehension Type

Paragraph for Q.No. 52 - 54

A parabola is drawn through two given points $A(1, 0)$ and $B(-1, 0)$ such that its directrix always touches the circle $x^2 + y^2 = 4$. Then

52. The equation of directrix is of the form

(a) $x \cos\alpha + y \sin\alpha = 1$ (b) $x \cos\alpha + y \sin\alpha = 2$

(c) $x \cos\alpha + y \sin\alpha = 3$ (d) $x \tan\alpha + y \sec\alpha = 2$

53. The locus of focus of the parabola is

(a) $\frac{x^2}{4} + \frac{y^2}{3} = 1$ (b) $\frac{x^2}{4} + \frac{y^2}{5} = 1$

(c) $\frac{x^2}{3} + \frac{y^2}{4} = 1$ (d) $\frac{x^2}{5} + \frac{y^2}{4} = 1$

54. The maximum possible length of semi latus rectum is

(a) $2 + \sqrt{3}$ (b) $3 + \sqrt{3}$

(c) $4 + \sqrt{3}$ (d) $1 + \sqrt{3}$

Paragraph for Q.No. 55 - 57

Let P, Q, R be three points on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and let P', Q', R' be their corresponding points on it's auxiliary circle, then

55. The maximum area of the triangle PQR is

(a) $\frac{3\sqrt{3}}{4}ab$ (b) $\frac{3\sqrt{3}}{2}ab$

(c) $\frac{\sqrt{3}}{4}ab$ (d) πab

56. $\frac{\text{Area of } \Delta PQR}{\text{Area of } \Delta P'Q'R'} =$

(a) $\frac{a}{b}$ (b) $\frac{b}{a}$

(c) $\frac{1}{2}$ (d) none of these

57. When the area of triangle PQR is maximum, the centroid of triangle $P'Q'R'$ lies at

(a) one focus (b) one vertex

(c) centre (d) on one directrix

Paragraph for Q.No. 58 - 60

Suppose an ellipse and a circle are respectively given by the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ...(i) and

$x^2 + y^2 + 2gx + 2fy + c = 0$ ...(ii)

The equation,

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right) + \lambda(x^2 + y^2 + 2gx + 2fy + c) = 0 \quad \text{...(iii)}$$

represents a curve which passes through the common points of the ellipse (i) and the circle (ii).

We can choose λ so that the equation (iii) represents a pair of straight lines. In general we get three values of λ, indicating three pair of straight lines can be drawn through the points. Also when (iii) represents a pair of straight lines they are parallel to the lines $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \lambda(x^2 + y^2) = 0$, which represents a pair of lines equally inclined to axes (the term containing y is absent). Hence two straight lines through the points

of intersection of an ellipse and any circle make equal angles with the axes.

58. The radius of the circle passing through the point of intersection of $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and $x^2-y^2=0$ is

(a) $\frac{ab}{\sqrt{a^2+b^2}}$ (b) $\frac{\sqrt{2}\,ab}{\sqrt{a^2+b^2}}$

(c) $\frac{a^2-b^2}{\sqrt{a^2+b^2}}$ (d) $\frac{a^2+b^2}{\sqrt{a^2+b^2}}$

59. Suppose two lines are drawn through the common points of intersection of $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and $x^2+y^2+2gx+2fy+c=0$. If these lines are inclined at an angle α, β to x-axis then

(a) $\alpha=\beta$ (b) $\alpha+\beta=\frac{\pi}{2}$

(c) $\alpha+\beta=\pi$ (d) $\alpha+\beta=2\tan^{-1}\left(\frac{b}{a}\right)$

60. The no. of pair of straight lines through the points of intersection of $x^2-y^2=1$ and $x^2+y^2-4x-5=0$ is

(a) 0 (b) 1

(c) 2 (d) 3

Matrix–Match Type

61. Match the following.

	Column-I		Column-II
(A)	The locus of mid-points of chords of an ellipse which are drawn through an end of minor axis, is	(p)	hyperbola
(B)	The locus of an end of latus rectum of all ellipses having a given major axis is	(q)	circle
(C)	The locus of the foot of perpendicular from a focus of the ellipse on any tangent is	(r)	parabola
(D)	A variable line is drawn through a fixed point cuts axes at A and B. The locus of the mid point of AB is	(s)	ellipse

Integer Answer Type

62. If e is the eccentricity of the hyperbola $(5x-10)^2+(5y+15)^2=(12x-5y+1)^2$ then $\frac{25e}{13}$ is equal to

63. The minimum distance of $4x^2+y^2+4x-4y+5=0$ from the line $-4x+3y=3$ is

64. The distance between the directrices of the ellipse $(4x-8)^2+16y^2=(x+\sqrt{3}y+10)^2$ is K then $\frac{K}{2}$ is

65. Number of points on the ellipse $\frac{x^2}{50}+\frac{y^2}{20}=1$ from which pair of perpendicular tangents are drawn to the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ is

66. If L be the length of common tangent to the ellipse $\frac{x^2}{25}+\frac{y^2}{4}=1$ and the circle $x^2+y^2=16$ intercepted by the coordinate axis then $\frac{\sqrt{3}L}{2}$ is

SOLUTIONS

1. **(c):** Given, equation of hyperbola is $x^2-3y^2=2x+8$

$\Rightarrow x^2-2x-3y^2=8$

$\Rightarrow (x-1)^2-3y^2=9 \Rightarrow \frac{(x-1)^2}{9}-\frac{y^2}{3}=1$

Conjugate of this hyperbola is $-\frac{(x-1)^2}{9}+\frac{y^2}{3}=1$

and its eccentricity $(e)=\sqrt{\left(\frac{a^2+b^2}{b^2}\right)}$

Here, $a^2=9, b^2=3 \therefore e=\sqrt{\frac{9+3}{3}}=2$

2. **(b) :** We know that equation of polar at point (h, k) is

$\frac{hx}{a^2}+\frac{ky}{b^2}=1 \Rightarrow \frac{hx}{4}+\frac{ky}{1}=1 \Rightarrow hx+4ky=4$ ...(i)

which is similar to given straight line $x+4y=4$ ...(ii)

Comparing (i) and (ii), we get $h=1$, $k=1$.

Hence, the point is (1, 1).

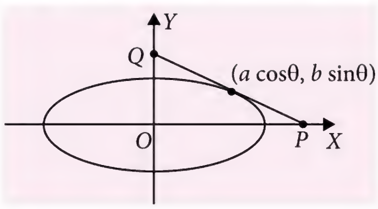
3. **(a) :** $y=\frac{-b}{a}x$ (Two diameters $y=m_1x$ and $y=m_2x$ will be conjugate diameters, if $m_1m_2=-\frac{b^2}{a^2}$).

4. **(c) :** Equation of tangent at $(a\cos\theta, b\sin\theta)$ is

$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$

$P=\left(\frac{a}{\cos\theta},0\right)$

$Q=\left(0,\frac{b}{\sin\theta}\right)$



Area of $\Delta OPQ = \frac{1}{2}\left|\left(\frac{a}{\cos\theta}\right)\left(\frac{b}{\sin\theta}\right)\right| = \frac{ab}{|\sin 2\theta|}$

$\therefore$ $(\text{Area})_{\min} = ab.$

5. (d) : $25(x-3)^2 + 16y^2 = 400$

$$\frac{(x-3)^2}{16} + \frac{y^2}{25} = 1 \quad \therefore \quad e = \sqrt{1 - \frac{16}{25}} = \frac{3}{5}$$

6. (c) : Since the general equation of second degree represents a rectangular hyperbola, if $\Delta \neq 0$, $h^2 > ab$ and coefficient of x^2 + coefficient of $y^2 = 0$. Therefore the given equation represents a rectangular hyperbola, if $\lambda + 5 = 0$ *i.e.*, $\lambda = -5$.

7. (c) : Multiplying both, we get

$$(bx)^2 - (ay)^2 = (ab)^2 \Rightarrow \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

which is the standard equation of hyperbola.

8. (b) : Since $e > 1$ always for hyperbola and $\frac{2}{3} < 1$.

9. (c) : We know that when a circle touches externally to the two given circles, then the locus of the circle will be hyperbola.

10. (c) : The equation of hyperbola is $\frac{x^2}{16} - \frac{y^2}{9} = 1$

Now $b^2 = a^2(e^2 - 1) \Rightarrow e = \frac{5}{4}$

Hence foci are $(\pm ae, 0) \Rightarrow \left(\pm 4 \cdot \frac{5}{4}, 0\right)$ *i.e.*, $(\pm 5, 0)$.

11. (c) : Since $h^2 > ab$

Hence it is a hyperbola.

12. (a) : The equation is $(x-0)^2 + (y-0)^2 = a^2$.

13. (c) : If $y = 2x + \lambda$ is tangent to given hyperbola, then

$$\lambda = \pm\sqrt{a^2m^2 - b^2} = \pm\sqrt{(100)(4) - 144} = \pm\sqrt{256} = \pm 16$$

14. (a) : Tangents to $\frac{x^2}{1} - \frac{y^2}{3} = 1$ and perpendicular to $x + 3y - 2 = 0$ are given by $y = 3x \pm \sqrt{9-3} = 3x \pm \sqrt{6}$.

15. (b) : Equation of hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Tangents to hyperbola are $y = mx \pm \sqrt{a^2m^2 - b^2}$ ...(i)

Tangents perpendicular to (i) are $y = \frac{-1}{m}x \pm \sqrt{\frac{a^2}{m^2} - b^2}$

Eliminating m, we get $x^2 + y^2 = a^2 - b^2$.

16. (b) : The tangent at (h, k) is $\frac{x}{4/h} - \frac{y}{3/k} = 1$

$\therefore \frac{4}{h} = \frac{3}{k} \Rightarrow \frac{h}{k} = \frac{4}{3}$ ...(i)

$3h^2 - 4k^2 = 12$ ...(ii)

Using (i) and (ii), we get the tangents as $y - x = \pm 1$.

17. (a) : The line through (6, 2) is

$y - 2 = m(x - 6) \Rightarrow y = mx + 2 - 6m$

Now from condition of tangency, $(2 - 6m)^2 = 25m^2 - 16$

$\Rightarrow 36m^2 + 4 - 24m - 25m^2 + 16 = 0$

$\Rightarrow 11m^2 - 24m + 20 = 0$

Obviously its roots are m_1 and m_2, therefore

$m_1 + m_2 = \frac{24}{11}$ and $m_1m_2 = \frac{20}{11}$

18. (a) : If $y = mx + c$ touches $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$,

then $c^2 = a^2m^2 - b^2$.

Here, $c = 6$, $a^2 = 100$, $b^2 = 49$

$\therefore 36 = 100m^2 - 49 \Rightarrow 100m^2 = 85 \Rightarrow m = \sqrt{\frac{17}{20}}$.

19. (c) : Equation of the tangent to $x^2 - y^2 - 8x + 2y + 11 = 0$ at (2, 1) is $2x - y - 4(x+2) + (y+1) + 11 = 0$ or $x = 2$.

20. (b) : $x\cos\alpha + y\sin\alpha = p \Rightarrow y = -\cot\alpha\, x + p\,\mathrm{cosec}\alpha$

It is tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Therefore, $p^2\,\mathrm{cosec}^2\alpha = a^2\cot^2\alpha - b^2$

$\Rightarrow a^2\cos^2\alpha - b^2\sin^2\alpha = p^2$.

21. (c) : We have, $x = 2\sec\phi$ and $y = 3\tan\phi$

$\Rightarrow \frac{dx}{d\phi} = 2\sec\phi\tan\phi$ and $\frac{dy}{d\phi} = 3\sec^2\phi$

$\therefore \frac{dy}{dx} = \frac{dy/d\phi}{dx/d\phi} = \frac{3\sec^2\phi}{2\sec\phi\tan\phi}$

$\frac{dy}{dx} = \frac{3}{2}\mathrm{cosec}\phi$ ...(i)

But, tangent is parallel to $3x - y + 4 = 0$ ...(ii)

By (i) and (ii), $\frac{3}{2}\mathrm{cosec}\phi = 3 \Rightarrow \mathrm{cosec}\phi = 2 \therefore \phi = 30°$.

22. (b) : Given equation of hyperbola is $xy = a$

Slope of tangent at point (x_1, y_1) is

$m = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} \quad \therefore \frac{xdy}{dx} + y = 0 \Rightarrow \frac{dy}{dx} = \frac{-y}{x}$

At point $(a, 1)$, we have $m = \left(\frac{dy}{dx}\right)_{(a,1)} = -\frac{1}{a}$

23. (d) : The condition for the line $y = mx + c$ will touch the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ if $c^2 = a^2m^2 - b^2$

Here, $m=-1, c=\sqrt{2}p,\ a^2=9, b^2=4$

$\therefore$ We get $2p^2 = 5$

24. (d) : The given ellipse is $\frac{x^2}{9}+\frac{y^2}{4}=1$. The value of the expression $\frac{x^2}{9}+\frac{y^2}{4}-1$ is positive for $x = 1, y = 2$ and negative for $x = 2, y = 1$. Therefore P lies outside E and Q lies inside E. The value of the expression $x^2 + y^2 - 9$ is negative for both the points P and Q. Therefore P and Q both lie inside C. Hence, P lies inside C but outside E.

25. (c) : We have, $a=6, b=2\sqrt{5}$

$\therefore\quad b^2=a^2(1-e^2)\Rightarrow \frac{20}{36}=(1-e^2)\Rightarrow e=\sqrt{\frac{16}{36}}=\frac{2}{3}$

But directrices are $x=\pm\frac{a}{e}$

Hence, distance between them is $2\cdot\frac{6}{2/3}=18$.

26. (a) : Vertices $\equiv (\pm 5, 0) \equiv (\pm a, 0) \Rightarrow a = 5$

Foci $(\pm 4,0)\equiv(\pm ae,0)\ \Rightarrow\ e=\frac{4}{5}$

$\therefore\quad b^2=(25)\left(1-\frac{16}{25}\right)=9\ \Rightarrow\ b = 9$

Hence equation is $\frac{x^2}{25}+\frac{y^2}{9}=1$ *i.e.*, $9x^2 + 25y^2 = 225$

27. (b) : We have, vertex $= (0,7)\ \Rightarrow\ b = 7$

Directrix $y = 12 \Rightarrow \frac{b}{e}=12\ \Rightarrow\ e=\frac{7}{12}$

Also, $a=7\sqrt{\frac{95}{144}}=\frac{7\sqrt{95}}{12}$

Hence equation of ellipse is $144x^2 + 95y^2 = 4655$.

28. (b) : $\frac{x^2}{4}+\frac{y^2}{3}=1$. Latus rectum $=\frac{2b^2}{a}=3$.

29. (b) : The ellipse is $\frac{x^2}{25}+\frac{y^2}{9}=1$

$\therefore\ e=\sqrt{1-\frac{b^2}{a^2}}=\sqrt{1-\frac{9}{25}}=\frac{4}{5}$

30. (a) : Since, $ae=\pm 2\ \Rightarrow\ a=\pm 4\quad (\because\ e=1/2)$

Now $b^2 = a^2(1 - e^2) \Rightarrow b^2 = 16(1 - 1/4) \Rightarrow b^2 = 12$

Hence ellipse is $\frac{x^2}{16}+\frac{y^2}{12}=1\ \Rightarrow\ 3x^2+4y^2=48$

31. (d) : We have, $2ae=8,\ \frac{2a}{e}=18\ \Rightarrow\ a=\sqrt{4\times 9}=6$

$e=\frac{2}{3},\ b=6\sqrt{1-\frac{4}{9}}=\frac{6}{3}\sqrt{5}=2\sqrt{5}$

Hence the required equation is $\frac{x^2}{36}+\frac{y^2}{20}=1$

i.e., $5x^2 + 9y^2 = 180$.

32. (b) : $4(x - 2)^2 + 9(y - 3)^2 = 36$

Hence, the centre is (2, 3).

33. (a) : Major axis $= 6 = 2a \Rightarrow a = 3$

Now, $e=\frac{1}{2}\ \Rightarrow b=3\sqrt{1-\frac{1}{4}}=\frac{3\sqrt{3}}{2}$

Also centre is (7, 0)

Equation is $\frac{(x-7)^2}{9}+\frac{y^2}{(27/4)}=1$

$\Rightarrow\ 3x^2 + 4y^2 - 42x + 120 = 0$

34. (b) : Given equation can be written as

$\frac{(x-1)^2}{5}+\frac{(y-2)^2}{9}=1\ \therefore\ e=\sqrt{\frac{b^2-a^2}{b^2}}=\sqrt{\frac{9-5}{9}}=\frac{2}{3}$

35. (c) : $E \equiv 4 + 9(3)^2 - 16(1) - 54(3) + 61 < 0$

Therefore, the point is inside the ellipse.

$\frac{4(x-2)^2}{36}+\frac{9(y-3)^2}{36}=1$

Equation of major axis is $y - 3 = 0$ and point (1, 3) lies on it.

36. (c) : It is the director circle.

37. (c) : Change the equation $9x^2 + 5y^2 - 30y = 0$ in standard form $9x^2 + 5(y^2 - 6y) = 0$

$\Rightarrow\ 9x^2+5(y^2-6y+9)=45\Rightarrow \frac{x^2}{5}+\frac{(y-3)^2}{9}=1$

$\because\ \ a^2 < b^2$, so axis of ellipse is on y-axis.

At y axis, put $x = 0$, so we get

$0 + 5y^2 - 30y = 0\quad \Rightarrow\quad y = 0, y = 6$

i.e., tangents at vertex is $y = 0, y = 6$.

38. (c) : As we know that the line $lx + my + n = 0$ is normal to $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, if $\frac{a^2}{l^2}+\frac{b^2}{m^2}=\frac{(a^2-b^2)^2}{n^2}$. But in this condition, we have to replace l by m, m by -1 and n by c, then the required condition is $c=\pm\frac{(a^2-b^2)m}{\sqrt{a^2+b^2m^2}}$.

39. (b) : $\frac{x-x_1}{x_1/a^2}=\frac{y-y_1}{y_1/b^2}$, which is the equation of normal at point (x_1, y_1).

In the given ellipse, $a^2 = 20, b^2 = \frac{180}{16}$

Hence the equation of normal at the point (2, 3) is

$\frac{x-2}{2/20} = \frac{y-3}{48/180} \Rightarrow 40(x-2) = 15(y-3)$

$\Rightarrow$ $8x - 3y = 7 \Rightarrow 3y - 8x + 7 = 0$.

40. (c) : Vertices $(\pm 4, 0) \equiv (\pm a, 0) \Rightarrow a = 4$

Foci $(\pm 6, 0) \equiv (\pm ae, 0)$ $\Rightarrow$ $e = \frac{6}{4} = \frac{3}{2}$

41. (c) : Foci $= (0, \pm 4) \equiv (0, \pm be) \Rightarrow be = 4$

Vertices $(= 0, \pm 2) \equiv (0, \pm b) \Rightarrow b = 2 \Rightarrow a = 2\sqrt{3}$

Hence equation is $\frac{-x^2}{(2\sqrt{3})^2} + \frac{y^2}{(2)^2} = 1$ or $\frac{y^2}{4} - \frac{x^2}{12} = 1$

42. (c) : Given equation of hyperbola, $\frac{x^2}{4} - \frac{y^2}{(16/9)} = 1$,

$\therefore$ $a = 2, b = \frac{4}{3}$. As we know, $b^2 = a^2(e^2 - 1)$

$\Rightarrow$ $\frac{16}{9} = 4(e^2 - 1) \Rightarrow e^2 = \frac{13}{9}$ $\therefore$ $e = \frac{\sqrt{13}}{3}$

43. (d): $\frac{x^2}{(6/2)} - \frac{y^2}{6} = 1 \Rightarrow a^2 = 3$ and $b^2 = 6$

Therefore $e = \sqrt{\frac{b^2}{a^2} + 1} \Rightarrow e = \sqrt{3}$

44. (c) : $\Delta \neq 0$, $h^2 > ab$

45. (b): $2ae = 16, e = \sqrt{2} \Rightarrow a = 4\sqrt{2}$ and $b = 4\sqrt{2}$

$\therefore$ Equation is $\frac{x^2}{(4\sqrt{2})^2} - \frac{y^2}{(4\sqrt{2})^2} = 1 \Rightarrow x^2 - y^2 = 32$

46. (c) : Hyperbola is $\frac{x^2}{144} - \frac{y^2}{81} = \frac{1}{25}$

$a = \sqrt{\frac{144}{25}}, b = \sqrt{\frac{81}{25}}, e_1 = \sqrt{1 + \frac{81}{144}} = \sqrt{\frac{225}{144}} = \frac{15}{12} = \frac{5}{4}$

Therefore, foci $= (\pm ae_1, 0) = \left(\pm \frac{12}{5} \cdot \frac{5}{4}, 0\right) = (\pm 3, 0)$

Also, focus of ellipse $= (4e, 0)$.

$\Rightarrow$ $e = \frac{3}{4}$. Hence, $b^2 = 16\left(1 - \frac{9}{16}\right) = 7$

47. (b): Here $a = b$, so it is a rectangular hyperbola. Hence, eccentricity, $e = \sqrt{2}$.

48. (c) : Foci of $xy = 36$ are $(c\sqrt{2}, c\sqrt{2})$ and $(-c\sqrt{2}, -c\sqrt{2}) = (6\sqrt{2}, 6\sqrt{2}), (-6\sqrt{2}, -6\sqrt{2})$

$\therefore$ Statement-1 is true.

The foci of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are $\left(\pm\sqrt{a^2 + b^2}, 0\right)$

$\therefore$ Statement-2 is false.

49. (c) : The point of intersection is, $\left(\sqrt{\frac{a^2 b}{a+b}}, \sqrt{\frac{ab^2}{a+b}}\right)$,

with $m_1 = \frac{-b^2}{a^2}\sqrt{\frac{a}{b}}, m_2 = -\sqrt{\frac{a}{b}}$.

$\Rightarrow \theta = \tan^{-1}\left|\left(\frac{m_1 - m_2}{1 + m_1 m_2}\right)\right| = \tan^{-1}\left(\frac{b-a}{\sqrt{ab}}\right)$.

50. (a) : Let the mid point $(t, b - t)$.

$\frac{tx}{2a^2} + \frac{(b-t)y}{2b^2} = \frac{t^2}{2a^2} + \frac{(b-t)^2}{2b^2}$

It passes through $(a, -b)$.

$\frac{ta}{2a^2} - \frac{b(b-t)}{2b^2} = \frac{t^2}{2a^2} + \frac{(b-t)^2}{2b^2}$

$\Rightarrow$ $t^2(a^2 + b^2) - ab(3a + b)t + 2a^2b^2 = 0$

For real t, $a^2b^2(3a + b)^2 - 4(a^2 + b^2)2a^2b^2 > 0$

$9a^2 + 6ab + b^2 - 8a^2 - 8b^2 > 0$

$a^2 + 6ab - 7b^2 > 0$

51. (a) : Given equation of ellipse is

$\frac{x^2}{32} + \frac{(y-12)^2}{64} = 1$. The maximum value of $\sqrt{x^2 + y^2}$ is the distance between (0, 0) & (0, 20).

52. (b) **53. (a)** **54. (a)**

52 - 54 : Any point on circle $x^2 + y^2 = 4$ is $(2\cos\alpha, 2\sin\alpha)$

$\therefore$ Equation of directrix is $x(\cos\alpha) + y(\sin\alpha) - 2 = 0$

Let focus be (x_1, y_1). Then as $A(1, 0)$, $B(-1, 0)$ lie on parabola we must have

$$\left.\begin{array}{l}(x_1 - 1)^2 + y_1^2 = (\cos\alpha - 2)^2 \\ (x_1 + 1)^2 + y_1^2 = (\cos\alpha + 2)^2\end{array}\right\} \Rightarrow x_1 = 2\cos\alpha,\ y_1 = \pm\sqrt{3}\sin\alpha$$

$\therefore$ Locus of focus is $\frac{x^2}{4} + \frac{y^2}{3} = 1$ and focus is of the form $(2\cos\alpha, \pm\sqrt{3}\sin\alpha)$.

$\therefore$ Length of semi latus rectum of parabola

$= \perp^r$ distance from focus to directrix $|2 \pm \sqrt{3}|\sin^2\alpha$.

Hence maximum possible length $= 2 + \sqrt{3}$

55. (a) : Let $P = (a\cos\alpha, b\sin\alpha)$, $P' = (a\cos\alpha, a\sin\alpha)$
$Q = (a\cos\beta, b\sin\beta)$, $Q' = (a\cos\beta, a\sin\beta)$
$R = (a\cos\gamma, b\sin\gamma)$, $R' = (a\cos\gamma, a\sin\gamma)$.

Area of ΔPQR is $2ab\sin\frac{\alpha-\beta}{2}\sin\frac{\beta-\gamma}{2}\sin\frac{\gamma-\alpha}{2}$

Its max value is $2ab\cdot\left(\frac{\sqrt{3}}{2}\right)(\sqrt{3}/2)(\sqrt{3}/2)=\frac{3\sqrt{3}ab}{4}$

56. (b) : $\frac{\text{Area of }\Delta PQR}{\text{Area of }\Delta P'Q'R'}$

$$=\frac{\frac{1}{2}ab\begin{vmatrix}\cos\alpha-\cos\gamma & \sin\alpha-\sin\gamma\\ \cos\alpha-\cos\beta & \sin\alpha-\sin\beta\end{vmatrix}}{\frac{1}{2}a^2\begin{vmatrix}\cos\alpha-\cos\gamma & \sin\alpha-\sin\gamma\\ \cos\alpha-\cos\beta & \sin\alpha-\sin\beta\end{vmatrix}}=\frac{b}{a}$$

57. (c) : Area of ΔPQR is max when $\alpha-\beta=\beta-\gamma=\gamma-\alpha=120°$ and $\Delta P'Q'R'$ is equilateral hence its centroid is (0, 0) centre of the ellipse.

58. (b) : $x^2+y^2=\frac{2a^2b^2}{a^2+b^2}$

$\therefore$ Radius of the circle $=\sqrt{\frac{2a^2b^2}{a^2+b^2}}=\frac{\sqrt{2}ab}{\sqrt{a^2+b^2}}$

59. (c) : As the lines joining common point of intersection must be equally inclined to the axis

$\tan\alpha=-\tan\beta\Rightarrow\alpha+\beta=n\pi$

60. (c) : Any curve through their point of intersection is $x^2+y^2-4x-5+\lambda(x^2-y^2-1)=0$

$\Rightarrow\ (1+\lambda)x^2+(1-\lambda)y^2-4x-5-\lambda=0$

Curve will be pair of straight lines, if

$abc+2fgh-af^2-by^2-ch^2=0$

$\Rightarrow\ (1+\lambda)(1-\lambda)(-5-\lambda)+0-(1+\lambda)\cdot0-(1-\lambda)\cdot4+(5+\lambda)\cdot0=0$

$\Rightarrow\ (\lambda-1)(\lambda+3)^2=0\Rightarrow\lambda=1,-3$

$\therefore$ Two pair of straight lines can be drawn.

61. (A) → (s), (B) → (r), (C) → (q), (D) → (p)

(A) The chord with mid point (h, k)

$\frac{hx}{a^2}+\frac{ky}{b^2}=\frac{h^2}{a^2}+\frac{k^2}{b^2}$ $\therefore$ Locus is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{y}{b}$

(B) (h, k) be the end of the latus rectum.

$h=ae,\ k=a(1-e^2)$

$h^2=-a(k-a)\Rightarrow x^2=-a(y-a)$, parabola.

(C) $y-mx=\pm\sqrt{a^2m^2+b^2}$ $\therefore$ $x^2+y^2=a^2$

(D) Line $\frac{x}{a}+\frac{y}{b}=1$ passes through fixed point (α,β)

$\therefore$ $\frac{\alpha}{a}+\frac{\beta}{b}=1$

$h=\frac{a}{2},k=\frac{b}{2}$

$\frac{\alpha}{x}+\frac{\beta}{y}=2\Rightarrow\left(x-\frac{\alpha}{2}\right)\left(y-\frac{\beta}{2}\right)=\frac{\alpha\beta}{4}$

62. (5) : Equation can be rewritten as

$\sqrt{(x-2)^2+(y+3)^2}=\frac{13}{5}\left|\frac{12x-5y+1}{13}\right|$ So, $e=\frac{13}{5}$.

63. (1) : The given curve represents the point $\left(-\frac{1}{2},2\right)$

$\therefore$ Minimum distance = 1.

64. (8) : $(x-2)^2+y^2=\left(\frac{1}{2}\right)^2\frac{(x+\sqrt{3}y+10)^2}{4}\Rightarrow e=1/2$

Perpendicular distance from (2, 0) to $x+\sqrt{3}y+10=0$ is $\frac{a}{e}-ae\Rightarrow 2a-\frac{a}{2}=6\Rightarrow a=4$

Distance between directrices $=\frac{2a}{e}=16=K$

65. (4) : Director circle of $\frac{x^2}{16}+\frac{y^2}{9}=1$ is $x^2+y^2=25$

The director circle will cut the ellipse $\frac{x^2}{50}+\frac{y^2}{20}=1$ at 4 points.

66. (7) : The equation of the tangent at $(5\cos\theta, 2\sin\theta)$ is $\frac{x}{5}\cos\theta+\frac{y}{2}\sin\theta=1$

If it is a tangent to the circle then $\frac{1}{\sqrt{\frac{\cos^2\theta}{25}+\frac{\sin^2\theta}{4}}}=4$

$\Rightarrow\ \cos\theta=\frac{10}{4\sqrt{7}},\sin\theta=\frac{\sqrt{3}}{2\sqrt{7}}$

Let A and B be the points where the tangent meets the coordinate axis then

$A\left(\frac{5}{\cos\theta},0\right)$, $B\left(0,\frac{2}{\sin\theta}\right)$

$L=\sqrt{\frac{25}{\cos^2\theta}+\frac{4}{\sin^2\theta}}=\frac{14}{\sqrt{3}}$

Solution Sender of Maths Musing

SET-180

- N. Jayanthi (Hyderabad)

CLASS XI | Series 9

ACE YOUR WAY CBSE

Statistics | Probability

IMPORTANT FORMULAE

STATISTICS

- *Range = Maximum value – Minimum value*
- *Mean,* $\bar{x} = A + \frac{\left(\sum_{i=1}^{n} f_i d_i\right)}{N} \times h$

  Where A = assumed Mean, h = width of class intervals and $d_i = \frac{x_i - A}{h}$
- *Mean deviation for ungrouped data*

  $M.D.(\bar{x}) = \frac{\Sigma|x_i - \bar{x}|}{n}$, $M.D.(M) = \frac{\Sigma|x_i - M|}{n}$
- *Mean deviation for grouped data*

  $M.D.(\bar{x}) = \frac{\Sigma f_i |x_i - \bar{x}|}{N}$, $M.D.(M) = \frac{\Sigma f_i |x_i - M|}{N}$,

  where $N = \Sigma f_i$
- *Variance and standard deviation of a discrete frequency distribution*

  $\sigma^2 = \frac{1}{N}\Sigma(x_i - \bar{x})^2$, $\sigma = \sqrt{\frac{1}{N}\Sigma(x_i - \bar{x})^2}$
- *Variance and standard deviation of a continuous frequency distribution*

  $\sigma^2 = \frac{1}{N}\Sigma f_i(x_i - \bar{x})^2$, $\sigma = \frac{1}{N}\sqrt{N\Sigma f_i x_i^2 - (\Sigma f_i x_i)^2}$
- *Shortcut method to find variance and standard deviation.*

  $\sigma^2 = \frac{h^2}{N^2}\left[N\Sigma f_i y_i^2 - (\Sigma f_i y_i)^2\right]$,

  $\sigma = \frac{h}{N}\sqrt{N\Sigma f_i y_i^2 - (\Sigma f_i y_i)^2}$, *where* $y_i = \frac{x_i - A}{h}$,

  A = assumed mean, h = width of class intervals
- *Coefficient of Variation (C.V.)* $= \frac{\sigma}{\bar{x}} \times 100, \bar{x} \neq 0$.

  For series with equal means, the series with lesser standard deviation is more consistent or less scattered.

PROBABILITY

- *Event A or B* $= A \cup B = \{x : x \in A \text{ or } x \in B\}$
- *Event A and B* $= A \cap B = \{x : x \in A \text{ and } x \in B\}$
- *Set* $A' = S - A$ *where S = sample space*
- $A - B$ = *Event 'A but not B'* $= A \cap B'$
- *For disjoint sets,* $A \cap B = \phi$
- *Number* $P(\omega_i)$ *associated with sample point* ω_i *such that*

  (i) $0 \leq P(\omega_i) \leq 1$

  (ii) $\Sigma P(\omega_i) = 1 \ \forall \ \omega_i \in S$

  (iii) $P(A) = \Sigma P(\omega_i) \ \forall \ \omega_i \in A$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$
- $P(A') = 1 - P(A)$

WORK IT OUT

VERY SHORT ANSWER TYPE

1. Calculate the mean deviation about median from the following data: 340, 150, 210, 240, 300, 310, 320.
2. There are 4 red and 3 black balls in a bag. If one ball is taken out of this bag at random, then represent the sample space and the event of this ball being black.
3. Two dice are thrown simultaneously. What is the probability of obtaining a total score of 7?
4. For a sample of size 60, we have the information $\sum x_i^2 = 18000$ and $\sum x_i = 960$. Find the variance.
5. What is the probability of getting a total of less than 12 in the throw of two dice?

SHORT ANSWER TYPE

6. The mean and variance of 7 observations are 8 and 16 respectively. If five of the observations are 2, 4, 10, 12, 14, find the remaining observations.
7. If a number of two digits is formed with the digits 2, 3, 5, 7, 9, without repetition of digits, what is the probability that the number formed is 35?
8. Three coins are tossed together. Find the probability of getting
   (i) at least two heads.
   (ii) at least one head and one tail.
9. The probability that a student will receive A, B, C or D grade are 0.35, 0.45, 0.09 and 0.11 respectively. Find the probability that the student will receive
   (i) A or B grade
   (ii) at most a C grade.
10. In a race, the odds in favour of horses P, Q, R, S are 1 : 2, 1 : 3, 1 : 4 and 1 : 5 respectively. Find the probability that one of them wins the race.

LONG ANSWER TYPE - I

11. Four cards are drawn at random from a pack of 52 playing cards. Find the probability of getting
    (i) all the four cards of the same suit
    (ii) all the four cards of the same number.
12. Calculate the mean deviation from the median of the following data:

Wages per week (in ₹)	10-20	20-30	30-40	40-50	50-60	60-70	70-80
No. of workers	4	6	10	20	10	6	4

13. The following is the record of goals scored by team A in football session.

No. of goals scored by team A	0	1	2	3	4
No. of matches	1	9	7	5	3

For team B, the number of goals scored per match was 2.5, with a S.D. of 1.25. Find which team may be considered more consistent.

14. What is the probability that in a group of 3 people at least two will have the same birthday? Assume that there are 365 days in a year and no one has his/her birthday on the 29$^{\text{th}}$ of February.
15. A box contains 4 red, 5 white and 6 black balls. A person draws 4 balls from the box at random. Find the probability of selecting at least one ball of each colour.

LONG ANSWER TYPE - II

16. Let $x_1, x_2, x_3, \ldots, x_n$ be n values of a variable X, and let $x_i = a + hu_i$, $i = 1, 2, \ldots, n$, where $u_1, u_2, \ldots, u_n$ are the values of variable U. Then, prove that Var $(X) = h^2$ Var (U), $h \neq 0$.
17. There are n articles to be distributed among N people. Find the probability of a particular person getting $r(< n)$ articles.
18. Find the mean and standard deviation of the first n terms of an A.P. whose first term is a and common difference is d.
19. Find the mean and variance of the data 1, 2, 2, 3, 3, 3, 4, 4, 4, 4, ... n, n, n,.........., n where the number r occurs r times, $r = 1, 2, 3, \ldots, n$.
20. The mean and standard deviation of 20 observations are found to be 10 and 2 respectively. On rechecking, it was found that an observation 8 was incorrect. Calculate the correct mean and standard deviation in each of the following cases :
    (i) If the wrong item is omitted.
    (ii) If it is replaced by 12.

SOLUTIONS

1. Arranging the observations in ascending order of magnitude, we obtain 150, 210, 240, 300, 310, 320, 340. Clearly, the middle observation is 300. So, median = 300.

x_i	$\lvert d_i\rvert = \lvert x_i - 300\rvert$
340	40
150	150
210	90
240	60
300	0
310	10
320	20
	$\Sigma\lvert d_i\rvert = \Sigma\lvert x_i - 300\rvert = 370$

$\therefore$ M.D.$(M)=\frac{1}{n}\sum|d_i|=\frac{1}{7}\sum|x_i-300|=\frac{370}{7}=52.8$

2. Let R_i denotes red balls and B_i denotes black balls.
S = Set of all possible outcomes when one ball is drawn from the bag = $\{R_1, R_2, R_3, R_4, B_1, B_2, B_3\}$
and E = Event of ball being black = $\{B_1, B_2, B_3\}$

3. Let S be the sample space and E be the event of "obtaining a total of 7."
Then, $n(S) = 6 \times 6 = 36$
Also, $E = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$
$\therefore\ n(E) = 6$
Now, required probability, $P(E)=\frac{n(E)}{n(S)}=\frac{6}{36}=\frac{1}{6}.$

4. Since, $\sigma^2=\frac{\sum x_i^2}{60}-\left(\frac{\sum x_i}{60}\right)^2$

$$=\frac{18000}{60}-\left(\frac{960}{60}\right)^2=300-256=44.$$

5. Let S be the sample space and E be the event that the sum of numbers coming up is 12
Then, $n(S) = 6 \times 6 = 36$
and $E = \{(6, 6)\}$ $\therefore\ n(E) = 1$
$\therefore\ P(E)=\frac{n(E)}{n(S)}=\frac{1}{36}$
$\therefore$ Required probability, $P(E')=1-P(E)=\frac{35}{36}$

6. Let a and b be the two observations.

$$\bar{x}=8 \Rightarrow \frac{a+b+2+4+10+12+14}{7}=8$$

$\therefore\ a + b = 56 - 42 = 14$

$$\sigma^2=16 \Rightarrow \frac{a^2+b^2+2^2+4^2+10^2+12^2+14^2}{7}-8^2=16$$

$\therefore\ a^2 + b^2 = 7(64 + 16) - 460 = 100$
$a + b = 14,\ a^2 + b^2 = 100$
$\Rightarrow\ a = 8, b = 6$ or $a = 6, b = 8$

7. Let S = the sample space
and E = the event that the number formed is 35
Now $n(E) = 1$ and $n(S)$ = total number of numbers of two digits formed with the digits 2, 3, 5, 7, 9 without repetition = ${}^5P_2 = 5 \times 4 = 20$
$\therefore$ Required probability, $P(E)=\frac{n(E)}{n(S)}=\frac{1}{20}$

8. Let S be the sample space for the given experiment.
A = the event of getting "at least two heads"
B = the event of getting "at least one head and one tail"
Now $S = \{(H, H, H), (H, H, T), (H, T, H), (T, H, H), (T, H, T), (T, T, H), (H, T, T), (T, T, T)\}$
$\therefore\ n(S) = 8$

(i) $A = \{(H, H, H), (H, H, T), (H, T, H), (T, H, H)\}$
$\therefore$ Required probability, $P(A)=\frac{n(A)}{n(S)}=\frac{4}{8}=\frac{1}{2}.$

(ii) $B = \{(H, H, T), (H, T, H), (T, H, H), (H, T, T), (T, H, T), (T, T, H)\}$
$\therefore$ Required probability, $P(B)=\frac{n(B)}{n(S)}=\frac{6}{8}=\frac{3}{4}.$

9. Let E_1, E_2, E_3 and E_4 denote the events of the student receiving A, B, C and D grades respectively.
Then, $P(E_1) = 0.35$, $P(E_2) = 0.45$, $P(E_3) = 0.09$ and $P(E_4) = 0.11$

(i) Required probability = $P(A \text{ or } B \text{ grade})$
$= P(E_1 \cup E_2)$
$= P(E_1) + P(E_2) = 0.35 + 0.45$
$= 0.80$

(ii) Required probability = $P(\text{at most a } C \text{ grade})$
$= P(C \text{ or } D \text{ grade})$
$= P(E_3 \cup E_4) = P(E_3) + P(E_4)$
$= 0.09 + 0.11 = 0.20$

10. Let A, B, C and D be the events that the horses P, Q, R and S respectively win the race.

Then, $P(A)=\frac{1}{3}$, $P(B)=\frac{1}{4}$, $P(C)=\frac{1}{5}$ and $P(D)=\frac{1}{6}$

Since there can be only one winner,
$\therefore$ A, B, C and D are mutually exclusive events

MPP-9 CLASS XI ANSWER KEY

1. (c)	2. (b)	3. (c)	4. (c)	5. (d)
6. (b)	7. (a,c)	8. (c,d)	9. (a,b,d)	10. (a,b,c)
11. (a,b,c,d)	12. (c,d)	13. (a,d)	14. (b)	15. (d)
16. (c)	17. (2)	18. (3)	19. (2)	20. (2)

Now, required probability = $P(A \cup B \cup C \cup D)$

$= P(A) + P(B) + P(C) + P(D)$

$= \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} = \frac{19}{20}$

11. Let S = the sample space

A = event that all the four cards are of the same suit.

and B = event that all the four cards have the same number

$n(S)$ = total number of ways of drawing 4 cards from a pack of 52 cards = ${}^{52}C_4$

(i) The four suits are club, space, heart and diamond, each having 13 cards.

Now, $n(A)$ = number of ways of getting all the four cards of the same suit

$= {}^{13}C_4 + {}^{13}C_4 + {}^{13}C_4 + {}^{13}C_4 = 4({}^{13}C_4)$

$\therefore$ Required probability,

$$P(A) = \frac{n(A)}{n(S)} = \frac{4({}^{13}C_4)}{{}^{52}C_4} = \frac{44}{4165}$$

(ii) Four cards bearing the same number can be drawn in the following ways :

(2, 2, 2, 2), ..., (10, 10, 10, 10),

(A, A, A, A), (J, J, J, J), (Q, Q, Q, Q), (K, K, K, K)

$\therefore$ $n(B)$ = number of favourable cases = 13

$\therefore$ Required probability, $P(B) = \frac{n(B)}{n(S)} = \frac{13}{{}^{52}C_4}$

$= \frac{1}{20825}$

12.

Wages per week (in ₹)	Mid-Values (x_i)	Frequency	Cummulative frequency	$\lvert d_i\rvert = \lvert x_i - 45\rvert$	$f_i\lvert d_i\rvert$
10-20	15	4	4	30	120
20-30	25	6	10	20	120
30-40	35	10	20	10	100
40-50	45	20	40	0	0
50-60	55	10	50	10	100
60-70	65	6	56	20	120
70-80	75	4	60	30	120
		$N = \Sigma f_i = 60$			$\Sigma f_i \lvert d_i\rvert = 680$

Here, N = 60. So, $\frac{N}{2} = 30$. The cumulative frequency just greater than $\frac{N}{2} = 30$ is 40 and the corresponding class is 40-50. So, 40-50 is the median class.

$\therefore$ $l = 40, f = 20, h = 10, C = 20.$

So, Median $= l + \frac{N/2 - C}{f} \times h = 40 + \frac{30-20}{20} \times 10 = 45.$

Thus, we have

$\Sigma f_i|x_i - 45| = \Sigma f_i|d_i| = 680$ and $N = 60$.

$\therefore$ Mean deviation from median $= \frac{\Sigma f_i |d_i|}{N} = \frac{680}{60} = 11.33$

13. Let the assumed mean for team A = 2.

No. of goals by team A	No. of matches (f_i)	$d_i = x_i - A = x_i - 2$	$f_i d_i$	$f_i d_i^2$
0	1	−2	−2	4
1	9	−1	−9	9
2	7	0	0	0
3	5	1	5	5
4	3	2	6	12
	$\Sigma f_i = 25$		$\Sigma f_i d_i = 0$	$\Sigma f_i d_i^2 = 30$

$$\text{A.M.} = \bar{x} = A + \frac{\Sigma f_i d_i}{\Sigma f_i} = 2 + \frac{0}{25} = 2$$

$$\text{S.D.} = \sigma = \sqrt{\frac{\Sigma f_i d_i^2}{\Sigma f_i} - \left(\frac{\Sigma f_i d_i}{\Sigma f_i}\right)^2} = \sqrt{\frac{30}{25}} = \sqrt{\frac{6}{5}} = 1.1$$

For team A : Coefficient of variation

$$= \frac{\sigma}{\bar{x}} \times 100 = \frac{1.1}{2} \times 100 = 55$$

For team B : Coefficient of variation $= \frac{1.25}{2.5} \times 100 = 50$

Clearly 55 > 50. Hence team 'B' is more consistent.

14. Let S = the sample space

and E = the event that "at least two people have the same birthday"

$\therefore$ E' = all the three people have distinct birthdays

Then, $n(S)$ = number of ways in which three persons may have their birthdays = 365 × 365 × 365 = 365^3

and $n(E') = 365 \times 364 \times 363$

Now, $P(E') = \frac{n(E')}{n(S)} = \frac{365 \times 364 \times 363}{365^3} = \frac{364 \times 363}{365^2}$

Hence, required probability,

$$P(E) = 1 - P(E') = 1 - \frac{364 \times 363}{365^2}$$

15. Let S = the sample space

Selecting the balls such that there is at least one ball of each colour can be drawn in the following mutually exclusive ways :

(i) 1 red, 1 white and 2 black balls

(ii) 1 red, 2 white and 1 black balls

(iii) 2 red, 1 white and 1 black balls

Let A = event that 1 red, 1 white and 2 black balls are drawn

B = event that 1 red, 2 white and 1 black balls are drawn

C = event that 2 red, 1 white and 1 black balls are drawn

Here A, B and C are mutually exclusive events.

Hence required probability = $P(A \cup B \cup C)$

$= P(A) + P(B) + P(C)$

$$= \frac{{}^4C_1 \cdot {}^5C_1 \cdot {}^6C_2}{{}^{15}C_4} + \frac{{}^4C_1 \cdot {}^5C_2 \cdot {}^6C_1}{{}^{15}C_4} + \frac{{}^4C_2 \cdot {}^5C_1 \cdot {}^6C_1}{{}^{15}C_4} = \frac{48}{91}$$

16. We have,

$x_i = a + h\,u_i,\ i = 1, 2, \ldots, n$

$$\Rightarrow \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} (a + h\,u_i)$$

$$\Rightarrow \sum_{i=1}^{n} x_i = na + h\sum_{i=1}^{n} u_i$$

$$\Rightarrow \frac{1}{n}\sum_{i=1}^{n} x_i = a + h\left(\frac{1}{n}\sum_{i=1}^{n} u_i\right)$$

$$\Rightarrow \overline{X} = a + h\,\overline{U} \qquad \left[\because\ \overline{X} = \frac{1}{n}\sum_{i=1}^{n} x_i \text{ and } \overline{U} = \frac{1}{n}\sum_{i=1}^{n} u_i\right]$$

$\therefore\ x_i - \overline{X} = (a + h\,u_i) - (a + h\,\overline{U}),\ i = 1, 2, \ldots, n$

$\Rightarrow\ x_i - \overline{X} = h\,(u_i - \overline{U}),\ i = 1, 2, \ldots, n$

$\Rightarrow\ (x_i - \overline{X})^2 = h^2(u_i - \overline{U})^2,\ i = 1, 2, \ldots, n$

$$\Rightarrow \frac{1}{n}\Sigma(x_i - \overline{X})^2 = h^2\left\{\frac{1}{n}\sum_{i=1}^{n}(u_i - \overline{U})^2\right\}$$

$\Rightarrow$ Var $(X) = h^2$ Var (U).

17. Let S = the sample space

E = the event that a particular person gets $r(< n)$ articles.

Now, each article can be given to any one of the N people.

$\therefore$ Each article can be distributed in N ways.

$\therefore$ $n(S)$ = number of ways of distributing n articles among N people

$= N \times N \times \ldots \times N = N^n$.

r articles out of n articles can be given to a particular person in ${}^nC_r$ ways and the remaining $(n - r)$ articles can be distributed to remaining $(N - 1)$ people in $(N - 1)^{n-r}$ ways.

$\therefore$ $n(E)$ = number of ways in which a particular person gets r articles

$= {}^nC_r \times (N - 1)^{n-r}$

Hence, required probability,

$$P(E) = \frac{n(E)}{n(S)} = \frac{{}^nC_r \times (N-1)^{n-r}}{N^n}$$

18. Here the first n terms of the A.P. would be $a, a + d, a + 2d + \ldots + a + (n - 1)d$ respectively.

Let the assumed mean be a.

x_i	$d_i = x_i - a$	d_i^2
a	0	0
$a + d$	d	d^2
$a + 2d$	$2d$	$4d^2$
...	...	...
...	...	...
$a + (n - 1)d$	$(n - 1)d$	$(n - 1)^2 d^2$

Now, $\Sigma d_i = 0 + d + 2d + \ldots. + (n - 1)d$

$$= [1 + 2 + \ldots + (n-1)]d = \frac{n(n-1)d}{2}$$

and $\Sigma d_i^2 = 0 + d^2 + 4d^2 + \ldots. + (n - 1)^2d^2$

$= [1^2 + 2^2 + \ldots + (n - 1)^2]d^2$

$$= \frac{n(n-1)(2n-1)d^2}{6}$$

The actual mean, $\bar{x} = a + \dfrac{\Sigma d_i}{n} = a + \dfrac{(n-1)d}{2}$

Standard deviation, $\sigma = \sqrt{\dfrac{\Sigma d_i^2}{n} - \left(\dfrac{\Sigma d_i}{n}\right)^2}$

EXAM CORNER 2018

Exam	Date
VITEEE	4th April to 15th April
JEE Main	8th April (Offline), 15th & 16th April (Online)
SRMJEEE	16th April to 30th April
Karnataka CET	18th April & 19th April
WBJEE	22nd April
Kerala PET	23rd April & 24th April
AMU (Engg.)	29th April
COMEDK (Engg.)	13th May
BITSAT	16th May to 31st May
JEE Advanced	20th May
AIIMS	27th May

$$= \sqrt{\frac{(n-1)(2n-1)d^2}{6} - \left(\frac{n-1}{2}\right)^2 d^2}$$

$$= d\sqrt{\frac{(n-1)}{2}\left[\frac{2n-1}{3} - \frac{n-1}{2}\right]} = d\sqrt{\frac{(n-1)(n+1)}{12}}$$

$$= d\sqrt{\frac{n^2-1}{12}}$$

19. $\bar{x} = \dfrac{1+2+2+.....+n+n}{1+2+3+....+n}$

$$= \frac{1^2+2^2+3^2+...+n^2}{n(n+1)/2} = \frac{n(n+1)(2n+1)}{6n(n+1)/2} = \frac{2n+1}{3}$$

$$\sigma^2 = \frac{1\cdot 1^2 + 2\cdot 2^2 + ... + n\cdot n^2}{n(n+1)/2} - \left(\frac{2n+1}{3}\right)^2$$

$$= \frac{1^3+2^3+3^3+....+n^3}{n(n+1)/2} - \left(\frac{2n+1}{3}\right)^2$$

$$= \frac{n(n+1)}{2} - \frac{(2n+1)^2}{9} = \frac{n^2+n-2}{18} = \frac{(n-1)(n+2)}{18}$$

20. We have, $n = 20$, $\bar{x} = 10$ and $\sigma = 2$

$\therefore \quad \bar{x} = \dfrac{1}{n}\Sigma x_i \Rightarrow \Sigma x_i = n\bar{x} = 20 \times 10 = 200$

$\therefore$ Incorrect $\Sigma x_i = 200$ and $\sigma = 2 \Rightarrow \sigma^2 = 4$

$\Rightarrow \dfrac{1}{n}\Sigma x_i^2 - \bar{x}^2 = 4 \Rightarrow \dfrac{1}{20}\Sigma x_i^2 - 100 = 4$

$\Rightarrow \Sigma x_i^2 = 2080$

$\therefore$ Incorrect $\Sigma x_i^2 = 2080$

(i) If we omit the wrong item, 8 from the observations, then 19 observations are left

Correct $\Sigma x_i + 8 =$ incorrect Σx_i

$\therefore$ Correct $\Sigma x_i = 200 - 8 = 192$

$\therefore$ Correct mean $= \dfrac{192}{19} = 10.10$

and correct $\Sigma x_i^2 + 8^2 =$ incorrect Σx_i^2

$\therefore$ Correct $\Sigma x_i^2 = 2080 - 64 = 2016$

Correct variance $= \dfrac{1}{19}$ (correct Σx_i^2) $-$ (correct mean)^2

$$= \frac{2016}{19} - \left(\frac{192}{19}\right)^2 = \frac{1440}{361}$$

$\therefore$ Correct standard deviation $= \sqrt{\dfrac{1440}{361}} = 1.997$

(ii) If we replace the wrong item by 12, then

Incorrect $\Sigma x_i - 8 + 12 =$ correct Σx_i

$\therefore$ Correct $\Sigma x_i = 200 + 4 = 204$

and correct mean $= \dfrac{204}{20} = 10.2$

and incorrect $\Sigma x_i^2 - 8^2 + 12^2 =$ correct Σx_i^2

$\therefore$ Correct $\Sigma x_i^2 = 2080 - 8^2 + 12^2 = 2160$

Correct variance $= \dfrac{1}{20}$ (correct Σx_i^2) $-$ (correct mean)^2

$$= \frac{2160}{20} - \left(\frac{204}{20}\right)^2 = \frac{1584}{400}$$

$\therefore$ Correct standard deviation

$$= \sqrt{\frac{1584}{400}} = 1.9899$$

◈◈

MPP-9 MONTHLY Practice Problems

Class XI

This specially designed column enables students to self analyse their extent of understanding of specified chapters. Give yourself four marks for correct answer and deduct one mark for wrong answer. Self check table given at the end will help you to check your readiness.

Conic Sections

Total Marks : 80 **Time Taken : 60 Min.**

Only One Option Correct Type

1. Equation of the smallest circle passing through the centre of the circle $x^2 + y^2 + 2x + 4y + 4 = 0$ and touching the circle $x^2 + y^2 - 4x - 2y - 3 = 0$ externally is
(a) $x^2 + y^2 - x + y - 4 = 0$
(b) $x^2 + y^2 + x - y - 6 = 0$
(c) $x^2 + y^2 + x + 3y + 2 = 0$
(d) none of these

2. The equation of the parabola whose vertex and focus lie on the axis of x at distances a and a_1 from the origin respectively is
(a) $y^2 = 4(a_1 - a)X$ (b) $y^2 = 4(a_1 - a)(X - a)$
(c) $y^2 = 4(a_1 - a)(X - a_1)$ (d) none of these

3. The curve described parametrically by $x = t^2 + t + 1$, $y = t^2 - t + 1$ represents
(a) a pair of straight line (b) an ellipse
(c) a parabola (d) a hyperbola

4. If angle between the tangents at the end points of the chords of the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$ is 90°, then locus of the mid points of the chords is
(a) $x^2 + y^2 = 9\left(\frac{x^2}{9} - \frac{y^2}{4}\right)^2$
(b) $x^2 + y^2 = 4\left(\frac{x^2}{9} - \frac{y^2}{4}\right)^2$
(c) $x^2 + y^2 = 5\left(\frac{x^2}{9} - \frac{y^2}{4}\right)^2$ (d) none of these

5. If number of common tangents of the circles $x^2 + y^2 + 2x + 6y + 1 = 0$ and $x^2 + y^2 - 6x + k = 0$ is three, then value of k is.
(a) 2 (b) 4 (c) 6 (d) 5

6. Tangent is the drawn to the ellipse $\frac{x^2}{27} + y = 1$ at $(3\sqrt{3}\cos\theta, \sin\theta)\left(\text{where } \theta \in \left(0, \frac{\pi}{2}\right)\right)$. Then the value of θ such that sum of intercepts on axes made by this tangent is minimum is
(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{8}$ (d) $\frac{\pi}{4}$.

One or More Than One Option(s) Correct Type

7. Parabola $y^2 = 4x$ and the cirlce having it's centre at (6, 5) intersect at right angle. Possible point of intersection of these curves can be
(a) (9, 6) (b) $(2, \sqrt{8})$ (c) (4, 4) (d) $(3, 2\sqrt{3})$

8. If the tangent at the point $P(\theta)$ to the ellipse $16x^2 + 11y^2 = 256$ is also a tangent to the circle $x^2 + y^2 - 2x = 15$, then θ is equal to
(a) $\frac{2\pi}{3}$ (b) $\frac{4\pi}{3}$ (c) $\frac{5\pi}{3}$ (d) $\frac{\pi}{3}$

9. If foci of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ coincide with the foci of $\frac{x^2}{25} - \frac{y^2}{9} = 1$ and eccentricity of the hyperbola is 2, then
(a) $a^2 + b^2 = 16$
(b) there is no director circle of the hyperbola
(c) centre of the director circle is (0, 0)
(d) length of latusrectum of the hyperbola = 12

10. At a point P on the parabola $y^2 = 4ax$, tangent and normal are drawn. Tangent intersects the x-axis at Q and normal intersects the curve at R such that chord PR subtends an angle of 90° at its vertex. Then

(a) $PQ = 2a\sqrt{6}$ (b) $PR = 6a\sqrt{3}$

(c) area of $\Delta PQR = 18\sqrt{2}a^2$

(d) $PQ = 3a\sqrt{2}$

11. P and Q are two points on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ whose eccentric angles are differ by 90°, then

(a) locus of point of intersection of tangents at P and Q is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=2$

(b) locus of mid-point (P, Q) is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{1}{2}$

(c) product of slopes of OP and OQ where O is the centre is $\frac{-b^2}{a^2}$

(d) max. area of ΔOPQ is $\frac{1}{2}ab$

12. The director circle of a hyperbola is $x^2 + y^2 - 4y = 0$. One end of the major axis is (2, 0) then the focus is

(a) $\left(\sqrt{3}, 2-\sqrt{3}\right)$ (b) $\left(-\sqrt{3}, 2+\sqrt{3}\right)$

(c) $\left(\sqrt{6}, 2-\sqrt{6}\right)$ (d) $\left(-\sqrt{6}, 2+\sqrt{6}\right)$

13. If the ellipse $x^2 + 2y^2 = 4$ and the hyperbola $S = 0$ have same end points of the latus rectum, then the eccentricity of the hyperbola can be

(a) $\operatorname{cosec}\frac{\pi}{4}$ (b) $\operatorname{cosec}\frac{\pi}{3}$

(c) $2\sin\frac{\pi}{3}+\sin\frac{\pi}{4}$ (d) $\sqrt{2}\sin\frac{\pi}{3}+\sin\frac{\pi}{4}$

Comprehension Type

Tangent to the parabola $y = x^2 + ax + 1$, at the point of intersection of y-axis also touches the circle $x^2 + y^2 = r^2$. Also, no point of the parabola is below the x-axis.

14. The radius of circle when a attains its maximum value

(a) $\frac{1}{\sqrt{10}}$ (b) $\frac{1}{\sqrt{5}}$ (c) 1 (d) $\sqrt{5}$

15. The minimum area bounded by the tangent and the coordinates axes is

(a) 1 (b) $\frac{1}{3}$ (c) $\frac{1}{2}$ (d) $\frac{1}{4}$

Matrix Match Type

16. Match the following.

	Column I		Column II
P.	The foci of the hyperbola $8x^2 - y^2 - 64x + 10y + 71 = 0$ are	1.	5
Q.	Director circles of $3x^2 + 2y^2 = 6$ and $3x^2 - 2y^2 = 6$ are $x^2 + y^2 = k$, where k is	2.	2, 3
R.	The foci of the hyperbola $9x^2 - 16y^2 - 36x + 96y + 36 = 0$ are	3.	(10, 5)
S.	Two circles $x^2 + y^2 + px + py - 7 = 0$ and $x^2 + y^2 - 10x + 2py + 1 = 0$ will orthogonally, then the value p is	4.	(2, 8)

	P	Q	R	S
(a)	2	1	3	4
(b)	1	3	4	2
(c)	3	1	4	2
(d)	3	4	1	2

Integer Answer Type

17. Number of distinct normal lines that can be drawn to ellipse $\frac{x^2}{169}+\frac{y^2}{25}=1$ from the point P (0, 6) is

18. From a point on $y = x + 1$ tangents are drawn to $\frac{x^2}{2}-y^2=1$ such that the chord of contact passes through fixed point (x_1, y_1), then $\frac{x_1+y_1}{y_1}$ is

19. If the normals of the parabola $y^2 = 4x$ drawn at the end points of its latus rectum are tangents to the circle $(x - 3)^2 + (y + 2)^2 = r^2$, then the value of r^2 is

20. The straight line $2x - 3y = 1$ divides the circular region $x^2 + y^2 \le 6$ into two parts.

If $S=\left\{\left(2,\frac{3}{4}\right),\left(\frac{5}{2},\frac{3}{4}\right),\left(\frac{1}{4},-\frac{1}{4}\right),\left(\frac{1}{8},\frac{1}{4}\right)\right\}$

then the number of points(s) in S lying inside the smaller part is

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SELF CHECK

No. of questions attempted ......

No. of questions correct ......

Marks scored in percentage ......

Check your score! If your score is

> 90%	EXCELLENT WORK !	You are well prepared to take the challenge of final exam.
90-75%	GOOD WORK !	You can score good in the final exam.
74-60%	SATISFACTORY !	You need to score more next time.
< 60%	NOT SATISFACTORY!	Revise thoroughly and strengthen your concepts.

CONCEPT MAP

PERMUTATIONS AND COMBINATIONS

Class XI

Properties

- ${}^nP_n = n! = n(n-1)\cdot(n-2) \ldots 3\cdot2\cdot1 = {}^nP_{n-1}$
- ${}^nP_1 = n$
- ${}^nP_0 = \dfrac{n!}{(n-0)!} = 1$
- ${}^nP_r = n\cdot{}^{n-1}P_{r-1} = n(n-1)\,{}^{n-2}P_{r-2} = n(n-1)(n-2)\,{}^{n-3}P_{r-3}$ and so on
- ${}^{n-1}P_r + r\cdot{}^{n-1}P_{r-1} = {}^nP_r$
- $\dfrac{{}^nP_r}{{}^nP_{r-1}} = n - r + 1$

Distributions into Groups : Distribution of n distinct things into r groups $G_1, G_2, \ldots, G_r$ containing $P_1, P_2, \ldots, P_r$ elements respectively.

- Groups are distinct : $\dfrac{n!}{P_1!\,P_2!\ldots P_r!}r!$
- Groups are identical : $\dfrac{n!}{P_1!\,P_2!\ldots P_r!}$

Number of selections of n objects

- Total number of selections of none or more objects from n distinct objects $= 2^n$.
- Number of selections from n distinct objects, taking at least one $= 2^n - 1$.
- Total number of selection of none or more objects from n identical objects $= (n+1)$.

Permutations

Arranging r objects out of n different things

- When repetition is not allowed $= {}^nP_r = \dfrac{n!}{(n-r)!}$, where $0 \le r \le n$
- When repetition is allowed $= n^r$

Combinations

Selecting r objects out of n different things

$= {}^nC_r = \dfrac{n!}{r!\,(n-r)!}, 0 \le r \le n$

Circular Permutations

Arrangement of n different things taken all at a time in form of circle is

- $(n-1)!$, if sense matter.
- $1/2(n-1)!$, if sense doesn't matter.

PERMUTATIONS AND COMBINATIONS

Properties

- ${}^nP_r = {}^nC_r\, r!, 0 \le r \le n$
- For $0 \le r \le n$, ${}^nC_r = {}^nC_{n-r}$
- For $1 \le r \le n$, ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$
- ${}^nC_a = {}^nC_b \Rightarrow a = b$ or $n = a + b$
- ${}^nC_0 + {}^nC_1 + \ldots\ldots + {}^nC_n = 2^n$

Factorial Notation

Product of first n natural numbers is denoted by $n!$ i.e., $n! = n(n-1)(n-2) \ldots 3.2.1$

Fundamental Principle of Counting

- **Addition Rule :** If a work is done only when any one of the number of works is done, then number of ways of doing that work is equal to the sum of number of ways of doing separate works.
- **Multiplication Rule :** If a work is done only when all of the number of works are done, then number of ways of doing that work is equal to the product of number of ways of doing separate works.

CONCEPT MAP

CONTINUITY AND DIFFERENTIABILITY

Class XII

CONTINUITY

A real valued function $f(x)$ is continuous at $x = a$ if $\lim\limits_{x\to a^-} f(x) = \lim\limits_{x\to a^+} f(x) = f(a)$

OR

In an open interval (a, b), $f(x)$ is continuous if it is continuous at every pt. between (a, b). In a closed interval $[a, b]$, $f(x)$ is continuous if

- $f(x)$ is continuous in (a, b)
- $\lim\limits_{x\to a^+} f(x) = f(a)$, $\lim\limits_{x\to b^-} f(x) = f(b)$

DISCONTINUITY AT A POINT

(i) Removable discontinuity : Discontinuity at $x = a$ $\lim\limits_{x\to a} f(x) \ne f(a)$

(ii) Discontinuity of 1st kind : $\lim\limits_{x\to a^-} f(x) \ne \lim\limits_{x\to a^+} f(x)$

(iii) Discontinuity of 2nd kind : Neither L.H.L. nor R.H.L. exists

Algebra of Continuous Functions

If f and g are two real functions continuous at a, then $f \pm g$, $f\cdot g$, f/g (where $g \ne 0$), kf (where $k \in R$), fog or gof are continuous at $x = a$.

Note : Constant, Polynomial, Modulus, logarithmic and exponential functions are everywhere continuous.

SOME IMPORTANT THEOREMS

Rolle's Theorem

If $f(x)$ is a real valued function defined on $[a, b]$ is continuous on $[a, b]$ and differential on (a, b) such that $f(a) = f(b)$, then $\exists$ atleast some $c \in (a, b)$ s.t. $f'(c) = 0$.

Lagrange's Mean Value Theorem

If $f(x)$ is a real valued function defined on $[a, b]$ is continuous on $[a, b]$ and differentiable on (a, b) then $\exists$ atleast some $c \in (a, b)$ s.t $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

DIFFERENTIABILITY

A real valued function $f(x)$ is differentiable at $x = c$ if $\lim\limits_{x\to c} \dfrac{f(x) - f(c)}{x - c}$ exists finitely or $Lf'(a) = Rf'(a)$

Note : If a function is differentiable at a point then it is continuous at that point. But the converse is not always true.

Algebra of Derivatives

- $(u \pm v)' = u' \pm v'$ • $(uv)' = u'v + uv'$ • $\left(\dfrac{u}{v}\right)' = \dfrac{u'v - uv'}{v^2}, v \ne 0$

Derivative of Functions

- Composite Function (Chain Rule)
  - Let $y = f(t)$ and $t = g(x)$, then $\dfrac{dy}{dx} = \dfrac{dy}{dt} \times \dfrac{dt}{dx}$.
  - Let $y = f(t)$, $t = g(u)$ and $u = m(x)$ then $\dfrac{dy}{dx} = \dfrac{dy}{dt} \times \dfrac{dt}{du} \times \dfrac{du}{dx}$.
- Implicit Function
  - Here, we differentiate the function of type $f(x, y) = 0$
- Logarithmic Function
  - If $y = u^v$, where u, v, are functions of x, then $\log y = v \log u \Rightarrow \dfrac{d}{dx}(u^v) = u^v\left[\dfrac{v}{u}\dfrac{du}{dx} + \log u \dfrac{dv}{dx}\right]$
- Parametric Function
  - If $x = f(t)$ and $y = g(t)$ then, $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{g'(t)}{f'(t)}, f'(t) \ne 0$

Second Order Derivative

- Let $y = f(x)$, then $\dfrac{dy}{dx} = f'(x)$

  If $f'(x)$ is differentiable, then $\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = f''(x)$ or $\dfrac{d^2y}{dx^2} = f''(x)$

Some Standard Derivatives

- $\dfrac{d}{dx}(\sin^{-1} x) = \dfrac{1}{\sqrt{1 - x^2}}$ • $\dfrac{d}{dx}(\cos^{-1} x) = \dfrac{-1}{\sqrt{1 - x^2}}$
- $\dfrac{d}{dx}(\tan^{-1} x) = \dfrac{1}{1 + x^2}$ • $\dfrac{d}{dx}(\cot^{-1} x) = \dfrac{-1}{1 + x^2}$
- $\dfrac{d}{dx}(\sec^{-1} x) = \dfrac{1}{x\sqrt{x^2 - 1}}$ • $\dfrac{d}{dx}(\text{cosec}^{-1} x) = \dfrac{-1}{x\sqrt{x^2 - 1}}$
- $\dfrac{d}{dx}(e^x) = e^x$ • $\dfrac{d}{dx}(\log x) = \dfrac{1}{x}$

CONCEPT BOOSTERS

Trigonometry

Class XII

This column is aimed at Class XII students so that they can prepare for competitive exams such as JEE Main/Advanced, etc. and be also in command of what is being covered in their school as part of NCERT syllabus. The problems here are a happy blend of the straight and the twisted, the simple and the difficult and the easy and the challenging.

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TRIGONOMETRIC FUNCTIONS

- An equation involving one or more trigonometric ratios of an unknown angle is called a trigonometric equation.
- The trigonometric equation may have infinite number of solutions (because of their periodic nature) and can be classified as :
  - Principal solution
  - General solution
- If $\sin\theta = \sin\alpha \Rightarrow \theta = n\pi + (-1)^n \alpha, n \in Z$
- If $\cos\theta = \cos\alpha \Rightarrow \theta = 2n\pi \pm \alpha, n \in Z$
- If $\tan\theta = \tan\alpha \Rightarrow \theta = n\pi + \alpha, n \in Z$
- If $\sin^2\theta = \sin^2\alpha \Rightarrow \theta = n\pi \pm \alpha, n \in Z$
- If $\cos^2\theta = \cos^2\alpha \Rightarrow \theta = n\pi \pm \alpha, n \in Z$
- If $\tan^2\theta = \tan^2\alpha \Rightarrow \theta = n\pi \pm \alpha, n \in Z$
- $\sin\theta = 0 \Rightarrow \theta = n\pi, n \in Z$
- $\cos\theta = 0 \Rightarrow \theta = (2n+1)\frac{\pi}{2}, n \in Z$
- $\tan\theta = 0 \Rightarrow \theta = n\pi, n \in Z$
- Trigonometric equations of the form $P(\sin x \pm \cos x, \sin x \cos x) = 0$, where $p(y, z)$ is a polynomial, can be solved by using the substitution $\sin x \pm \cos x = t$
- Trigonometric equations of the form $a \sin x + b \cos x = c$, where $a, b, c \in R$, can be solved by dividing both sides of the equation by $\sqrt{a^2 + b^2}$
- Trigonometric equations can also be solved by transforming a product of trigonometric ratios into their sum or difference and viceversa.

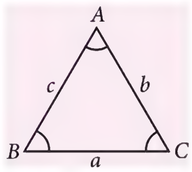
Application of Trigonometric Functions

Sine Formula

- In any triangle ABC, the sines of the angles are proportional to the opposite sides

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

where R is circumradius of the ΔABC.



Cosine Formula

- $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$
- $\cos B = \dfrac{c^2 + a^2 - b^2}{2ca}$
- $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

Projection Formula

- $a = b \cos C + c \cos B$
- $b = c \cos A + a \cos C$
- $c = a \cos B + b \cos A$

Napolian or Napier's Analogy

- $\tan\left(\dfrac{A-B}{2}\right) = \dfrac{a-b}{a+b}\cot\dfrac{C}{2}$
- $\tan\left(\dfrac{B-C}{2}\right) = \dfrac{b-c}{b+c}\cot\dfrac{A}{2}$
- $\tan\left(\dfrac{C-A}{2}\right) = \dfrac{c-a}{c+a}\cot\dfrac{B}{2}$

Half Angle Formula

- $\sin\dfrac{A}{2} = \sqrt{\dfrac{(s-b)(s-c)}{bc}}$

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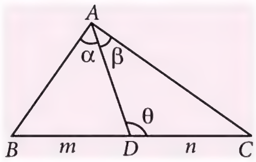
- $\cos\dfrac{A}{2} = \sqrt{\dfrac{s(s-a)}{bc}}$
- $\tan\dfrac{A}{2} = \sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}}$

where $s = \dfrac{a+b+c}{2}$ is semi perimetre of triangle.

- Area of the ΔABC



$(\Delta) = \dfrac{1}{2}ab\sin C$

$= \dfrac{1}{2}bc\sin A = \dfrac{1}{2}ca\sin B$

- $(m+n)\cot\theta = m\cot\alpha - n\cot\beta$
- $R = \dfrac{a}{2\sin A} = \dfrac{b}{2\sin B} = \dfrac{c}{2\sin C} = \dfrac{abc}{4\Delta}$
- $r = \dfrac{\Delta}{s}$, where r is in radius of ΔABC.
- $r = (s-a)\tan\dfrac{A}{2} = (s-b)\tan\dfrac{B}{2} = (s-c)\tan\dfrac{C}{2}$
- $r = \dfrac{a\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}}$
- $r = 4R\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}$

If r_1, r_2 and r_3 are ex radius of ΔABC respectively then,

- $r_1 = \dfrac{\Delta}{s-a} = s\tan\dfrac{A}{2}$; $r_2 = \dfrac{\Delta}{s-b} = s\tan\dfrac{B}{2}$

$r_3 = \dfrac{\Delta}{s-c} = s\tan\dfrac{C}{2}$

- Length of an angle bisector from the angle A,

$\beta_a = \dfrac{2bc\cos\frac{A}{2}}{b+c}$

- Length of median from the angle A

$m_a = \dfrac{1}{2}\sqrt{2b^2+2c^2-a^2}$

Also, $m_a^2 + m_b^2 + m_c^2 = \dfrac{3}{4}(a^2+b^2+c^2)$

- Length of altitude from the angle $A = \dfrac{2\Delta}{a}$

INVERSE TRIGONOMETRIC FUNCTIONS

- If $f: A \to B$ such that $f(x) = y$ is one-one and onto, then there exists a unique function f^{-1}, where $f^{-1}: B \to A$ and $f^{-1}(y) = x$, for all $x \in X$, $y \in Y$.

i.e., domain of f^{-1} = range of f and range of f^{-1} = domain of f

- **Domain and Range of Inverse Trigonometrical Functin s**

Functin s	Dm ä n	Ra g
$\sin^{-1}x$	$[-1, 1]$	$[-\pi/2, \pi/2]$
$\cos^{-1}x$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1}x$	$(-\infty, \infty)$	$(-\pi/2, \pi/2)$
$\cot^{-1}x$	$(-\infty, \infty)$	$(0, \pi)$
$\sec^{-1}x$	$(-\infty, -1] \cup [1, \infty)$	$[0, \pi/2) \cup (\pi/2, \pi]$
$\text{cosec}^{-1}x$	$(-\infty, -1] \cup [1, \infty)$	$[-\pi/2, 0) \cup (0, \pi/2]$

- $\sin^{-1}x + \sin^{-1}y = \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$, $-1 \le x, y \le 1$, $x^2 + y^2 \le 1$
- $\sin^{-1}x - \sin^{-1}y = \sin^{-1}\left\{x\sqrt{1-y^2} - y\sqrt{1-x^2}\right\}$, $-1 \le x, y \le 1$ and $x^2 + y^2 \le 1$
- $\cos^{-1}x + \cos^{-1}y = \cos^{-1}\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right)$, $-1 \le x, y \le 1$, $x + y \ge 0$
- $\cos^{-1}x - \cos^{-1}y = \cos^{-1}\left\{xy + \sqrt{1-x^2}\sqrt{1-y^2}\right\}$, $-1 \le x, y \le 1$ and $x - y \le 0$
- $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\dfrac{x+y}{1-xy}\right)$, $xy < 1$
- $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\dfrac{x-y}{1+xy}\right)$, $xy > -1$
- $2\tan^{-1}x = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$, $|x| \le 1$

$= \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right), x \ge 0 = \tan^{-1}\left(\dfrac{2x}{1-x^2}\right), -1 < x < 1$

- $2\sin^{-1}x = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$, $-\dfrac{1}{\sqrt{2}} \le x \le \dfrac{1}{\sqrt{2}}$

- $2\cos^{-1}x = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$, $\frac{1}{\sqrt{2}} \le x \le 1$
- $3\sin^{-1}x = \sin^{-1}(3x - 4x^3)$, $-\frac{1}{2} \le x \le \frac{1}{2}$
- $3\cos^{-1}x = \cos^{-1}(4x^3 - 3x)$, $\frac{1}{2} \le x \le 1$
- $3\tan^{-1}x = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$, $-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$

PROBLEMS

Single Correct Answer Type

1. General solution of the equation $\cot\theta - \tan\theta = 2$ is

(a) $n\pi + \frac{\pi}{4}$ (b) $\frac{n\pi}{2} + \frac{\pi}{8}$

(c) $\frac{n\pi}{4} \pm \frac{\pi}{8}$ (d) none of these

2. If $\cos\theta + \sec\theta = \frac{5}{2}$, then the general value of θ is

(a) $n\pi \pm \frac{\pi}{3}, n \in Z$ (b) $2n\pi \pm \frac{\pi}{6}, n \in Z$

(c) $n\pi \pm \frac{\pi}{6}, n \in Z$ (d) $2n\pi \pm \frac{\pi}{3}, n \in Z$

3. If $\tan^2\theta - (1+\sqrt{3})\tan\theta + \sqrt{3} = 0$, then the general value of θ (where $n \in Z$) is

(a) $n\pi + \frac{\pi}{4}, n\pi + \frac{\pi}{3}$ (b) $n\pi - \frac{\pi}{4}, n\pi + \frac{\pi}{3}$

(c) $n\pi + \frac{\pi}{4}, n\pi - \frac{\pi}{3}$ (d) $n\pi - \frac{\pi}{4}, n\pi - \frac{\pi}{3}$

4. If $\cos 2\theta + 3\cos\theta = 0$, then the general value of θ is

(a) $2n\pi \pm \cos^{-1}\left(\frac{-3+\sqrt{17}}{4}\right), n \in Z$

(b) $2n\pi \pm \cos^{-1}\left(\frac{-3-\sqrt{17}}{4}\right), n \in Z$

(c) $n\pi \pm \cos^{-1}\left(\frac{-3+\sqrt{17}}{4}\right), n \in Z$

(d) $n\pi \pm \cos^{-1}\left(\frac{-3-\sqrt{17}}{4}\right), n \in Z$

5. If $\tan\theta - \sqrt{2}\sec\theta = \sqrt{3}$, then the general value of θ (where $n \in Z$) is

(a) $n\pi + (-1)^n \frac{\pi}{4} - \frac{\pi}{3}$ (b) $n\pi + (-1)^n \frac{\pi}{3} - \frac{\pi}{4}$

(c) $n\pi + (-1)^n \frac{\pi}{3} + \frac{\pi}{4}$ (d) $n\pi + (-1)^n \frac{\pi}{4} + \frac{\pi}{3}$

6. If $\tan\theta + \tan 2\theta + \tan 3\theta = \tan\theta \tan 2\theta \tan 3\theta$, then the general value of θ is

(a) $n\pi, n \in Z$ (b) $\frac{n\pi}{6}, n \in Z$

(c) $n\pi \pm \frac{\pi}{3}, n \in Z$ (d) $\frac{n\pi}{2}, n \in Z$

7. If $\cos p\theta = \cos q\theta$, $p \ne q$, $n \in Z$ then

(a) $\theta = 2n\pi$ (b) $\theta = \frac{2n\pi}{p \pm q}$

(c) $\theta = \frac{n\pi}{p+q}$ (d) none of these

8. If $\sin 3\alpha = 4\sin\alpha \sin(x+\alpha)\sin(x-\alpha)$, then $x =$

(a) $n\pi \pm \frac{\pi}{6}, n \in Z$ (b) $n\pi \pm \frac{\pi}{3}, n \in Z$

(c) $n\pi \pm \frac{\pi}{4}, n \in Z$ (d) $n\pi \pm \frac{\pi}{2}, n \in Z$

9. The general solution of the equation $(\sqrt{3}-1)\sin\theta + (\sqrt{3}+1)\cos\theta = 2$ (where $n \in Z$) is

(a) $2n\pi \pm \frac{\pi}{4} + \frac{\pi}{12}$ (b) $n\pi + (-1)^n \frac{\pi}{4} + \frac{\pi}{12}$

(c) $2n\pi \pm \frac{\pi}{4} - \frac{\pi}{12}$ (d) $n\pi + (-1)^n \frac{\pi}{4} - \frac{\pi}{12}$

10. The general value θ is obtained from the equation $\cos 2\theta = \sin\alpha$ is

(a) $2\theta = \frac{\pi}{2} - \alpha$

(b) $\theta = 2n\pi \pm \left(\frac{\pi}{2} - \alpha\right), n \in Z$

(c) $\theta = \frac{n\pi + (-1)^n \alpha}{2}, n \in Z$

(d) $\theta = n\pi \pm \left(\frac{\pi}{4} - \frac{\alpha}{2}\right), n \in Z$

11. The equation $3\sin^2 x + 10\cos x - 6 = 0$ is satisfied, if

(a) $x = n\pi \pm \cos^{-1}(1/3)$ (b) $x = 2n\pi \pm \cos^{-1}(1/3)$

(c) $x = n\pi \pm \cos^{-1}(1/6)$ (d) $x = 2n\pi \pm \cos^{-1}(1/6)$

12. The general value of θ in the equation $2\sqrt{3}\cos\theta = \tan\theta$, is

(a) $2n\pi \pm \frac{\pi}{6}, n \in Z$ (b) $2n\pi \pm \frac{\pi}{4}, n \in Z$

(c) $n\pi + (-1)^n \frac{\pi}{3}, n \in Z$ (d) $n\pi + (-1)^n \frac{\pi}{4}, n \in Z$

13. The set of values of x for which the expression $\frac{\tan 3x - \tan 2x}{1 + \tan 3x \tan 2x} = 1$, is

(a) ϕ (b) $\pi/4$

(c) $\left\{n\pi+\frac{\pi}{4}: n=1,2,3.....\right\}$

(d) $\left\{2n\pi+\frac{\pi}{4}: n=1,2,3.....\right\}$

14. The equation $\sin x \cos x = 2$ has

(a) one solution (b) two solutions

(c) infinite solutions (d) no solutions

15. If $\sin 5x + \sin 3x + \sin x = 0$, then the value of x other than 0 lying between $0 \le x \le \frac{\pi}{2}$ is

(a) $\pi/6$ (b) $\pi/12$ (c) $\pi/3$ (d) $\pi/4$

16. The number of values of x in the interval $[0, 5\pi]$ satisfying the equation $3\sin^2 x - 7\sin x + 2 = 0$ is

(a) 0 (b) 5 (c) 6 (d) 10

17. The equation $\sin x + \sin y + \sin z = -3$ for $0 \le x \le 2\pi$, $0 \le y \le 2\pi$, $0 \le z \le 2\pi$ has

(a) one solution (b) two sets of solutions

(c) four sets of solutions

(d) no solution

18. If $2\sin^2\theta = 3\cos\theta$ where $0 \le \theta \le 2\pi$, then $\theta =$

(a) $\frac{\pi}{6}, \frac{7\pi}{6}$ (b) $\frac{\pi}{3}, \frac{5\pi}{3}$

(c) $\frac{\pi}{3}, \frac{7\pi}{3}$ (d) none of these

19. The domain of definition of the function $f(x) = \sin^{-1}(|x-1|-2)$ is

(a) $[-3, 0] \cup [1, 3]$ (b) $[-2, 0] \cup [1, 4]$

(c) $[-2, 0] \cup [2, 4]$ (d) $[-2, 0] \cup [1, 2]$

20. If $(2\cos x - 1)(3 + 2\cos x) = 0$, $0 \le x \le 2\pi$, then $x =$

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{3}, \frac{5\pi}{3}$

(c) $\frac{\pi}{2}, \frac{5\pi}{3}, \cos^{-1}\left(-\frac{3}{2}\right)$ (d) $\frac{5\pi}{3}$

21. If $\sin\theta = \sqrt{3}\cos\theta, -\pi < \theta < 0$, then $\theta =$

(a) $-\frac{5\pi}{6}$ (b) $-\frac{4\pi}{6}$ (c) $\frac{4\pi}{6}$ (d) $\frac{5\pi}{6}$

22. The most general value of θ which will satisfy both the equations $\sin\theta = -\frac{1}{2}$ and $\tan\theta = \frac{1}{\sqrt{3}}$ is

(a) $n\pi + (-1)^n \frac{\pi}{6}$ (b) $n\pi + \frac{\pi}{6}$

(c) $2n\pi \pm \frac{\pi}{3}$ (d) none of these

23. The smallest positive angle which satisfies the equation $2\sin^2\theta + \sqrt{3}\cos\theta + 1 = 0$, is

(a) $\frac{5\pi}{6}$ (b) $\frac{2\pi}{3}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{6}$

24. If $\cos 40° = x$ and $\cos\theta = 1 - 2x^2$, then the possible values of θ lying between 0° and 360° is

(a) 100° and 260° (b) 80° and 280°

(c) 280° and 100° (d) 110° and 260°

25. If $\tan(\pi\cos\theta) = \cot(\pi\sin\theta)$, then $\sin\left(\theta+\frac{\pi}{4}\right)$ equals

(a) $\frac{1}{\sqrt{2}}$ (b) $\frac{1}{2}$ (c) $\frac{1}{2\sqrt{2}}$ (d) $\frac{\sqrt{3}}{2}$

26. The general solution of $\sin x - \cos x = \sqrt{2}$, for any integer n is

(a) $n\pi$ (b) $2n\pi + \frac{3\pi}{4}$

(c) $2n\pi$ (d) $(2n + 1)\pi$

27. If $12\cot^2\theta - 31\operatorname{cosec}\theta + 32 = 0$, then the value of $\sin\theta$ is

(a) $\frac{3}{5}$ or 1 (b) $\frac{2}{3}$ or $\frac{-2}{3}$

(c) $\frac{4}{5}$ or $\frac{3}{4}$ (d) $\pm\frac{1}{2}$

28. Period of $\frac{\sin\theta+\sin 2\theta}{\cos\theta+\cos 2\theta}$ is

(a) 2π (b) π (c) $2\pi/3$ (d) $\pi/3$

29. Period of $\sin\theta - \sqrt{3}\cos\theta$ is

(a) $\pi/4$ (b) $\pi/2$ (c) π (d) 2π

30. The period of the function $\sin\left(\frac{2x}{3}\right)+\sin\left(\frac{3x}{2}\right)$ is

(a) 2π (b) 10π (c) 6π (d) 12π

31. Which of the following functions has period 2π?

(a) $y = \sin\left(2\pi t+\frac{\pi}{3}\right)+2\sin\left(3\pi t+\frac{\pi}{4}\right)+3\sin 5\pi t$

(b) $y = \sin\frac{\pi}{3}t + \sin\frac{\pi}{4}t$

(c) $y = \sin t + \cos 2t$

(d) none of these

32. The possible values of x, which satisfy the trigonometric equation $\tan^{-1}\left(\frac{x-1}{x-2}\right)+\tan^{-1}\left(\frac{x+1}{x+2}\right)=\frac{\pi}{4}$ are

(a) $\pm\frac{1}{\sqrt{2}}$ (b) $\pm\sqrt{2}$ (c) $\pm\frac{1}{2}$ (d) ± 2

33. If $\sin^{-1} x + \sin^{-1} y = \frac{2\pi}{3}$ and $\cos^{-1} x - \cos^{-1} y = \frac{\pi}{3}$ Then, (x, y) is equal to
(a) (0, 1) (b) (1/2, 1)
(c) (1, 1/2) (d) $(\sqrt{3}/2, 1)$

34. In ΔABC, $a\sin(B - C) + b\sin(C - A) + c\sin(A - B) =$
(a) 0 (b) $a + b + c$
(c) $a^2 + b^2 + c^2$ (d) $2(a^2 + b^2 + c^2)$

35. In triangle ABC, if a, b, c are in A.P., then the value of $\frac{\sin\frac{A}{2}\sin\frac{C}{2}}{\sin\frac{B}{2}} =$
(a) 1 (b) 1/2 (c) 2 (d) -1

36. In a ΔABC, if $\angle C = 30°$, $a = 47$ cm and $b = 94$ cm, then the triangle is
(a) Right angled (b) Right angled isosceles
(c) Isosceles (d) none of these

37. If $a = 9$, $b = 8$ and $c = x$ satisfies $3\cos C = 2$ then
(a) $x = 5$ (b) $x = 6$
(c) $x = 4$ (d) $x = 7$

38. If in a triangle, $a\cos^2\frac{C}{2} + c\cos^2\frac{A}{2} = \frac{3b}{2}$, then its sides will be in
(a) A.P. (b) G.P.
(c) H.P. (d) A.G.

39. In a triangle ABC if $2a^2b^2 + 2b^2c^2 = a^4 + b^4 + c^4$, then angle B is equal to
(a) 45° or 135° (b) 135° or 120°
(c) 30° or 60° (d) none of these

40. If $a = 2$, $b = 3$, $c = 5$ in ΔABC, then $\angle C =$
(a) $\pi/6$ (b) $\pi/3$
(c) $\pi/2$ (d) none of these

41. Point D, E are taken on the side BC of a triangle ABC such that $BD = DE = EC$. If $\angle BAD = x$, $\angle DAE = y$, $\angle EAC = z$, then the value of $\frac{\sin(x+y)\sin(y+z)}{\sin x \sin z} =$
(a) 1 (b) 2
(c) 4 (d) none of these

42. If a_1, a_2, a_3 are in A.P. and if d is the common difference, then $\tan^{-1}\left(\frac{d}{1+a_1a_2}\right) + \tan^{-1}\left(\frac{d}{1+a_2a_3}\right) =$
(a) $\tan^{-1}\left(\frac{2d}{1+a_1a_3}\right)$ (b) $\tan^{-1}\left(\frac{d}{1+a_1a_3}\right)$
(c) $\tan^{-1}\left(\frac{2d}{1+a_2a_3}\right)$ (d) $\tan^{-1}\left(\frac{2d}{1-a_1a_3}\right)$

43. Given $0 \le x \le \frac{1}{2}$ then the value of $\tan\left[\sin^{-1}\left\{\frac{x}{\sqrt{2}} + \frac{\sqrt{1-x^2}}{\sqrt{2}}\right\} - \sin^{-1} x\right]$ is
(a) 1 (b) $\sqrt{3}$ (c) -1 (d) $\frac{1}{\sqrt{3}}$

44. In a ΔABC, $a = 5$, $b = 4$ and $\cos(A - B) = \frac{31}{32}$, then side c is equal to
(a) 6 (b) 7
(c) 9 (d) none of these

45. The smallest angle of the triangle whose sides are $6+\sqrt{12}, \sqrt{48}, \sqrt{24}$ is
(a) $\pi/3$ (b) $\pi/4$
(c) $\pi/6$ (d) none of these

46. If angles of a triangle are in the ratio of 2 : 3 : 7, then the sides are in the ratio of
(a) $\sqrt{2}:2:(\sqrt{3}+1)$ (b) $2:\sqrt{2}:(\sqrt{3}+1)$
(c) $\sqrt{2}:(\sqrt{3}+1):2$ (d) $2:(\sqrt{3}+1):\sqrt{2}$

47. If sides of a triangle are 2 cm, $\sqrt{6}$ cm and $(\sqrt{3}+1)$ cm. Then, the smallest angle of the triangle is
(a) 30° (b) 45° (c) 60° (d) 75°

48. In any triangle ABC, $\frac{\tan\frac{A}{2} - \tan\frac{B}{2}}{\tan\frac{A}{2} + \tan\frac{B}{2}} =$
(a) $\frac{a-b}{a+b}$ (b) $\frac{a-b}{c}$
(c) $\frac{a-b}{a+b+c}$ (d) $\frac{c}{a+b}$

Assertion & Reason Type

Directions : *In the following questions, Statement-1 is followed by Statement-2. Mark the correct choice as :*

(a) Statement-1 is true, Statement-2 is true, Statement-2 is a correct explanation for Statement-1.
(b) Statement-1 is true, Statement-2 is true, Statement-2 is not a correct explanation for Statement-1.
(c) Statement-1 is true, Statement-2 is false.
(d) Statement-1 is false, Statement-2 is true.

49. **Statement-1:** The number of integral values of λ, for which the equation $7\cos x + 5\sin x = 2\lambda + 1$ has a solution, is 8.
Statement-2 : $a\cos\theta + b\sin\theta = c$ has atleast one solution if $|c| > \sqrt{a^2+b^2}$.

50. **Statement-1:** $\sin 2 > \sin 3$.

Statement-2 : If $x, y \in \left(\frac{\pi}{2}, \pi\right), x < y$, then $\sin x > \sin y$

51. Let $\alpha, \beta, \gamma > 0$ and $\alpha+\beta+\gamma=\frac{\pi}{2}$.

Statement-1: If $\left|\tan\alpha\tan\beta-\frac{a!}{6}\right|+\left|\tan\beta\tan\gamma-\frac{b!}{2}\right|$ $+\left|\tan\gamma\tan\alpha-\frac{c!}{3}\right|\le 0$, where $n! = 1\cdot 2 \ldots n)$

then $\tan\alpha\tan\beta$, $\tan\beta\tan\gamma$, $\tan\gamma\tan\alpha$ are in A.P.

Statement-2 : $\tan\alpha\tan\beta+\tan\beta\tan\gamma+\tan\gamma\tan\alpha=1$

52. ABC is an isosceles triangle. h_a is the length of the altitude from A. h_b has a similar meaning.
Given $a \le h_a$ and $b \le h_b$.

Statement-1 : $a : b : c = 1 : 1 : \sqrt{2}$

Statement-2 : In any triangle the length of an altitude does not exceed that of the corresponding median.

53. **Statement-1 :** In a ΔABC, if $a < b < c$ and r is inradius and r_1, r_2, r_3 are the exradii opposite to angle A, B, C respectively then $r < r_1 < r_2 < r_3$.

Statement-2: For, ΔABC, $r_1r_2+r_2r_3+r_3r_1=\frac{r_1r_2r_3}{r}$

54. Let a, b, c be 3 positive real numbers, such that $\sqrt[3]{\frac{a^3+b^3+c^3+3abc}{2}} > \max(a, b, c)$, then

Statement-1 : It is impossible to form a triangle whose sides have length a, b and c.

Statement-2 : $p > \max(a, b, c)$, $p > a$, $p > b$ and $p > c$.

55. **Statement-1 :** In a ΔABC, if $\cos A + 2\cos B + \cos C = 2$, then a, b, c are in A.P.

Statement-2 : In a ΔABC, we have $\cos A + \cos B + \cos C$

$$=1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \text{ and } \sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}},$$

$$\sin\frac{B}{2}=\sqrt{\frac{(s-c)(s-a)}{ca}} \quad \sin\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{ab}}$$

where $2s = a + b + c$

56. The angles of a right angled triangle ABC are in A.P.

Statement-1 : $\frac{r}{R}=\frac{\sqrt{3}-1}{2}$

Statement-2 : $\frac{r}{s}=\frac{2-\sqrt{3}}{\sqrt{3}}$

Comprehension Type

Paragraph for Q. No. 57 – 59

If curve of $y = f(x)$ and $y = g(x)$ intersects at n different points $x = x_1, x_2, x_3$ .... Then equation $f(x) = g(x)$ is said to have n solutions.

57. Number of solutions of $|\cos x| = 2[x]$ is (where $[x]$ is integral part of x)

(a) 0 (b) 1
(c) 2 (d) infinite

58. The number of solutions of $\sin\pi x = |\log_e|x||$ is

(a) 0 (b) 6 (c) 4 (d) 8

59. Number of solutions of the equation $\sin^5 x-\cos^5 x=\frac{1}{\cos x}-\frac{1}{\sin x}$ $(\sin x \ne \cos x)$ is

(a) 0 (b) 1
(c) 2 (d) infinite

Paragraph for Q. No. 60 – 62

The sides of a triangle ABC are 7, 8, 6 the smallest angle being 'C'.

60. The length of the altitude from vertex 'C' is

(a) $5\sqrt{3}$ (b) $\frac{\sqrt{35}}{4}$
(c) $\frac{7}{3}\sqrt{15}$ (d) $\frac{7}{4}\sqrt{15}$

61. The length of the median from vertex C is

(a) $\frac{\sqrt{95}}{4}$ (b) $\frac{\sqrt{95}}{2}$ (c) $\sqrt{\frac{95}{2}}$ (d) $\frac{\sqrt{95}}{3}$

62. The length of the internal bisector of angle C is

(a) $\sqrt{30}$ (b) $\frac{14}{5}\sqrt{6}$ (c) $\frac{14}{5}$ (d) $2\sqrt{6}$

Paragraph for Q. No. 63 – 65

Let ABC be any triangle and P be a point inside it such that $\angle PAB=\frac{\pi}{18}$, $\angle PBA=\frac{\pi}{9}$, $\angle PCA=\frac{\pi}{6}$, $\angle PAC=\frac{2\pi}{9}$.
Let $\angle PCB = x$.

63. $x =$

(a) $\pi/9$ (b) $2\pi/9$
(c) $\pi/3$ (d) none of these

64. ΔABC is

(a) equilateral (b) isosceles
(c) scalene (d) right angled

65. Which of the following is true?

(a) $BC > AC$ (b) $AC < AB$
(c) $AC > AB$ (d) $BC = AC$

Paragraph for Q. No. 66 – 68

$$\sum_{r=1}^{n} \tan^{-1}\left(\frac{x_r - x_{r-1}}{1+x_{r-1}x_r}\right) = \sum_{r=1}^{n} (\tan^{-1} x_r - \tan^{-1} x_{r-1})$$
$$= \tan^{-1}x_n - \tan^{-1}x_0,\ \forall\ n \in N$$

66. The value of
$\mathrm{cosec}^{-1}\sqrt{5} + \mathrm{cosec}^{-1}\sqrt{65} + \mathrm{cosec}^{-1}\sqrt{(325)} + \ldots$ to ∞ is

(a) π (b) $\frac{3\pi}{4}$ (c) $\frac{\pi}{2}$ (d) $\frac{\pi}{4}$

67. The sum to infinite terms of the series

$\cot^{-1}\left(2^2+\frac{1}{2}\right)+\cot^{-1}\left(2^3+\frac{1}{2^2}\right)+\cot^{-1}\left(2^4+\frac{1}{2^3}\right)+\ldots$ is

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$ (c) $\cot^{-1}2$ (d) $-\cot^{-1}2$

68. The sum to infinite terms of the series
$\cot^{-1}3 + \cot^{-1}7 + \cot^{-1}13 + \ldots$ is

(a) $\frac{\pi}{2}$ (b) $\cot^{-1}2$

(c) $\tan^{-1}2$ (d) $\frac{\pi}{4}$

Matrix-Match Type

69. Match the following.
If in ΔABC, $r = 2$, $r_1 = 4$, $s = 12$ and $a < b < c$, then

Column-I		Column-II	
(A)	Area of ΔABC is	(p)	22
(B)	$4 + 4R$	(q)	24
(C)	$\angle A$ of triangle is	(r)	$\sin^{-1}(4/5)$
(D)	$\angle B$ of triangle is	(s)	$\sin^{-1}(3/5)$

70. Match the following.

Column-I		Column-II	
(A)	If principal values of $\sin^{-1}\left(-\frac{1}{2}\right)+\tan^{-1}(\sqrt{3})$ and $\cos^{-1}\left(-\frac{1}{2}\right)$ are λ and μ respectively, then	(p)	$\lambda+\mu=\frac{\pi}{2}$
(B)	If principal values of $\sin^{-1}\left(\sin\frac{7\pi}{6}\right)$ and $\cos^{-1}\left\{-\sin\left(\frac{5\pi}{6}\right)\right\}$ are λ and μ respectively, then	(q)	$\lambda+\mu=\frac{5\pi}{6}$
(C)	If principal values of $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ and $\sin^{-1}\left\{\cos\left(\sin^{-1}\frac{\sqrt{3}}{2}\right)\right\}$ are λ and μ respectively, then	(r)	$\lambda+\mu=-\frac{\pi}{6}$
		(s)	$\mu-\lambda=\frac{5\pi}{6}$

Integer Answer Type

71. If $\theta = \cot^{-1} 7 + \cot^{-1} 8 + \cot^{-1}18$, then the value of $\cot\theta$ must be

72. If $\theta \in [0, 5\pi]$ and $r \in R$ such that $2\sin\theta = r^4 - 2r^2 + 3$ then the maximum no. of values of the pair (r, θ) is.

73. If $\lambda = \cos^4 [\tan^{-1} \{\sin (\cot^{-1}5)\}]$, then the value of $\frac{729}{169}\lambda$ must be

74. If $\sin^{-1} x + \sin^{-1}y = \pi$ and, if $x = \lambda y$, then the value of λ must be

75. The number of solutions of x, which satisfy the equation $\log_{|\sin x|} (1 + \cos x) = 2$ for $x \in [0, 2\pi]$ is

76. If $S=\sum_{r=1}^{50}\tan^{-1}\left(\frac{2r}{2+r^2+r^4}\right)$, then the value of $\frac{2550}{638}\cot S$ must be

77. In a ΔABC, $a = 5$, $b = 4$ and $\tan\frac{C}{2}=\sqrt{\frac{7}{9}}$, then measure of side '$c$' is

78. In triangle ABC, $a=\sqrt{5};\ b=2;\ \angle A=\frac{\pi}{6}$ and c_1 and c_2 are the two possible values of third side then $|c_1 - c_2|$ is

79. In ΔABC, $3a = b + c$ then $\cot B/2 \cot C/2$ is

80. In ΔABC, if $r_1 = 6$, $R = 5$, $r = 2$, then the value of $6\tan A$ is

SOLUTIONS

1. (b) : On simplification, given equation reduces to $\cos 2\theta = \sin 2\theta$

$$\Rightarrow \tan 2\theta = \tan\frac{\pi}{4} \Rightarrow 2\theta = n\pi+\frac{\pi}{4} \Rightarrow \theta = \frac{n\pi}{2}+\frac{\pi}{8}, n \in Z$$

2. (d) : $\cos^2\theta - \frac{5}{2}\cos\theta + 1 = 0$

$$\Rightarrow \cos\theta = \frac{(5/2)\pm\sqrt{(25/4)-4}}{2} = \frac{5\pm3}{4}$$

$\Rightarrow \cos\theta = \frac{1}{2} = \cos\left(\frac{\pi}{3}\right) \Rightarrow \theta = 2n\pi \pm \frac{\pi}{3}, n \in Z$

3. (a) : $\tan^2\theta - \tan\theta - \sqrt{3}\tan\theta + \sqrt{3} = 0$

$\Rightarrow \tan\theta(\tan\theta - 1) - \sqrt{3}(\tan\theta - 1) = 0$

$\Rightarrow (\tan\theta - \sqrt{3})(\tan\theta - 1) = 0 \Rightarrow \theta = n\pi + \frac{\pi}{3}, n\pi + \frac{\pi}{4}$

4. (a) **5. (d)**

6. (b) : Given, $\tan\theta + \tan 2\theta + \tan 3\theta = \tan\theta\tan 2\theta\tan 3\theta$

$$\tan 6\theta = \frac{\tan\theta + \tan 2\theta + \tan 3\theta - \tan\theta\tan 2\theta\tan 3\theta}{1 - \sum\tan\theta\tan 2\theta}$$

$= 0$, (from the given condition)

$\Rightarrow 6\theta = n\pi \Rightarrow \theta = n\pi/6, n \in Z$

7. (b) : $\cos p\theta = \cos q\theta \Rightarrow p\theta = 2n\pi \pm q\theta$

$\Rightarrow \theta = \frac{2n\pi}{p \pm q}, n \in Z$

8. (b) : $3\sin\alpha - 4\sin^3\alpha = 4\sin\alpha(\sin^2 x - \sin^2\alpha)$

$\therefore \sin^2 x = \left(\frac{\sqrt{3}}{2}\right)^2 \Rightarrow \sin^2 x = \sin^2 \pi/3$

$\Rightarrow x = n\pi \pm \pi/3, n \in Z$

9. (a)

10. (d) : $\cos 2\theta = \cos\left(\frac{\pi}{2} - \alpha\right) \Rightarrow 2\theta = 2n\pi \pm \left(\frac{\pi}{2} - \alpha\right)$

$\Rightarrow \theta = n\pi \pm \left(\frac{\pi}{4} - \frac{\alpha}{2}\right), n \in Z$

11. (b) : We have, $3\sin^2 x + 10\cos x - 6 = 0$

$\Rightarrow 3(1 - \cos^2 x) + 10\cos x - 6 = 0$

On solving, $(\cos x - 3)(3\cos x - 1) = 0$

Either $\cos x = 3$ (not possible)

or $\cos x = \frac{1}{3} \Rightarrow x = 2n\pi \pm \cos^{-1}(1/3), n \in Z$

12. (c)

13. (a) : $\tan(3x - 2x) = \tan x = 1 \Rightarrow x = n\pi + \frac{\pi}{4}, n \in Z$

But this value does not satisfy the given equation. Hence option (a) is the correct answer.

14. (d) : $\sin x\cos x = 2$ or $\sin 2x = 4$, which is impossible.

15. (c) : Given, $\sin 5x + \sin 3x + \sin x = 0$

$\Rightarrow -\sin 3x = \sin 5x + \sin x = 2\sin 3x\cos 2x$

$\Rightarrow \sin 3x = 0 \Rightarrow x = 0$

or $\cos 2x = -\frac{1}{2} = -\cos\left(\frac{\pi}{3}\right) = \cos\left(\pi - \frac{\pi}{3}\right)$

$\Rightarrow 2x = 2n\pi \pm \frac{2\pi}{3} \Rightarrow x = n\pi \pm \left(\frac{\pi}{3}\right)$

For x lying between 0 and $\frac{\pi}{2}$, we get $x = \frac{\pi}{3}$

16. (c)

17. (a) : Given, $\sin x + \sin y + \sin z = -3$ which is satisfied only when $x = y = z = \frac{3\pi}{2}$, for $x, y, z \in [0, 2\pi]$

18. (b) : $2 - 2\cos^2\theta = 3\cos\theta$

$\Rightarrow 2\cos^2\theta + 3\cos\theta - 2 = 0$

$\Rightarrow \cos\theta = \frac{-3 \pm \sqrt{9 + 16}}{4} = \frac{-3 \pm 5}{4}$

Neglecting (–) sign, we get

$\cos\theta = \frac{1}{2} = \cos\left(\frac{\pi}{3}\right) \Rightarrow \theta = 2n\pi \pm \frac{\pi}{3}.$

The values of θ between 0 and 2π are $\frac{\pi}{3}, \frac{5\pi}{3}$.

19. (c)

20. (b) : Given, $(2\cos x - 1)(3 + 2\cos x) = 0$

Then, $\cos x = \frac{1}{2}$ as $\cos x \neq \frac{-3}{2}$

$$\Rightarrow x = 2n\pi \pm \frac{\pi}{3}; \left\{\begin{array}{l} \text{for } n = 0, x = \frac{\pi}{3} \\ \text{for } n = 1, x = \frac{5\pi}{3} \end{array}\right\}$$

21. (b) : $\tan\theta = \sqrt{3} = \tan\frac{\pi}{3} \Rightarrow \theta = n\pi + \frac{\pi}{3}$

for $-\pi < \theta < 0$

Put $n = -1$, we get $\theta = -\pi + \frac{\pi}{3} = \frac{-2\pi}{3}$ or $\frac{-4\pi}{6}$

22. (d) : $\sin\theta = -\frac{1}{2} = \sin\left(-\frac{\pi}{6}\right) = \sin\left(\pi + \frac{\pi}{6}\right)$

$\tan\theta = \frac{1}{\sqrt{3}} = \tan\left(\frac{\pi}{6}\right) = \tan\left(\pi + \frac{\pi}{6}\right) \Rightarrow \theta = \left(\pi + \frac{\pi}{6}\right)$

Hence general value of θ is $2n\pi + \frac{7\pi}{6}$.

23. (a) : $2 - 2\cos^2\theta + \sqrt{3}\cos\theta + 1 = 0$

$\Rightarrow 2\cos^2\theta - \sqrt{3}\cos\theta - 3 = 0$

$\Rightarrow \cos\theta = \frac{\sqrt{3} \pm \sqrt{3 + 24}}{4} = \frac{\sqrt{3}(1 \pm 3)}{4} = \sqrt{3}\left(-\frac{1}{2}\right)$

$\Rightarrow \theta = \frac{5\pi}{6}$

24. (a) : Here, $\cos\theta = 1-2\cos^2 40° = -(2\cos^2 40° - 1)$
$= -\cos(2\times 40°) = -\cos 80°$
$= \cos(180°+80°) = \cos(180°-80°)$
Hence, $\cos 260°$ and $\cos 100°$ *i.e.*, $\theta = 100°$ and $260°$

25. (c) : $\tan(\pi\cos\theta) = \tan\left(\frac{\pi}{2} - \pi\sin\theta\right)$

$\therefore\ \sin\theta + \cos\theta = \frac{1}{2} \Rightarrow \sin\left(\theta + \frac{\pi}{4}\right) = \frac{1}{2\sqrt{2}}$

26. (b) : $\sin x\frac{1}{\sqrt{2}} - \cos x\frac{1}{\sqrt{2}} = 1 \Rightarrow \cos\left(x+\frac{\pi}{4}\right) = -1$

$\Rightarrow x + \frac{\pi}{4} = 2n\pi \pm \pi \Rightarrow 2n\pi + \frac{3\pi}{4}$ or $2n\pi - \frac{5\pi}{4}$

27. (c) : Given, $12\cot^2\theta - 31\operatorname{cosec}\theta + 32 = 0$
$\Rightarrow 12(\operatorname{cosec}^2\theta - 1) - 31\operatorname{cosec}\theta + 32 = 0$
$\Rightarrow 12\operatorname{cosec}^2\theta - 31\operatorname{cosec}\theta + 20 = 0$
$\Rightarrow 12\operatorname{cosec}^2\theta - 16\operatorname{cosec}\theta - 15\operatorname{cosec}\theta + 20 = 0$
$\Rightarrow (4\operatorname{cosec}\theta - 5)(3\operatorname{cosec}\theta - 4) = 0$

$\Rightarrow \operatorname{cosec}\theta = \frac{5}{4}, \frac{4}{3} \quad \therefore \sin\theta = \frac{4}{5}, \frac{3}{4}$

28. (c) : $\frac{\sin\theta + \sin 2\theta}{\cos\theta + \cos 2\theta} = \frac{2\sin\left(\frac{3\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}{2\cos\left(\frac{3\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)} = \tan\left(\frac{3\theta}{2}\right)$

Hence period $= \frac{2\pi}{3}$

29. (d) : $\sin\theta - \sqrt{3}\cos\theta = 2\sin\left(\theta - \frac{\pi}{3}\right)$
Hence period $= 2\pi$

30. (d) : Period of $\sin\left(\frac{2x}{3}\right) = \frac{2\pi}{2/3} = 3\pi$

Period of $\sin\left(\frac{3x}{2}\right) = \frac{2\pi}{3/2} = \frac{4\pi}{3}$

L.C.M. of 3π and $\frac{4\pi}{3} = 12\pi$. Hence period is 12π.

31. (c)

32. (a) : $\tan^{-1}\left(\frac{x+1}{x+2}\right) = \tan^{-1}1 - \tan^{-1}\left(\frac{x-1}{x-2}\right)$

$\Rightarrow \frac{x+1}{x+2} = \frac{1-\frac{x-1}{x-2}}{1+\frac{x-1}{x-2}} \Rightarrow \frac{x+1}{x+2} = \frac{-1}{2x-3}$

$\Rightarrow 2x^2 = 1 \Rightarrow x = \pm\frac{1}{\sqrt{2}}$

33. (b)

34. (a) : $a\sin(B-C) + b\sin(C-A) + c\sin(A-B)$
$= k(\Sigma\sin A\sin(B-C) = k\{\Sigma\sin(B+C)\sin(B-C)\}$

$= k\left\{\Sigma\frac{1}{2}(\cos 2C - \cos 2B)\right\} = 0$

35. (b) : $\frac{\sin\frac{A}{2}\sin\frac{C}{2}}{\sin\frac{B}{2}}$

$= \sqrt{\frac{ac(s-b)(s-c)(s-b)(s-a)}{(s-a)(s-c)bc\times ab}} = \frac{s-b}{b}$

But a, b and c are in A. P. $\Rightarrow 2b = a + c$

Hence, $\frac{s-b}{b} = \frac{\frac{3b}{2}-b}{b} = \frac{1}{2}$

36. (d) : $\cos C = \frac{a^2+b^2-c^2}{2ab}$

$\Rightarrow \frac{\sqrt{3}}{2} = \frac{(47)^2+(94)^2-c^2}{2\times 47\times 94} \Rightarrow c = 58.24$

37. (d) : $\cos C = \frac{2}{3} = \frac{81+64-x^2}{2\cdot 9\cdot 8}$

$\Rightarrow x^2 = 49 \Rightarrow x = 7$

38. (a) : $a\frac{s(s-c)}{ab} + c\frac{s(s-a)}{bc} = \frac{3b}{2}$

$\Rightarrow 2s(s-c+s-a) = 3b^2 \Rightarrow 2s(b) = 3b^2 \Rightarrow 2s = 3b$

$\Rightarrow a+b+c = 3b \Rightarrow a+c = 2b \Rightarrow a, b, c$ are in A.P.

39. (a) : $2a^2b^2 + 2b^2c^2 = a^4 + b^4 + c^4$
Also, $(a^2-b^2+c^2)^2 = a^4+b^4+c^4 - 2(a^2b^2+b^2c^2-c^2a^2)$

$\Rightarrow (a^2-b^2+c^2)^2 = 2c^2a^2$

$\Rightarrow \frac{a^2-b^2+c^2}{2ca} = \pm\frac{1}{\sqrt{2}} = \cos B$

$\Rightarrow \angle B = 45°$ or $135°$

40. (d) : $\cos C = \frac{a^2+b^2-c^2}{2ab} = -1$

$\Rightarrow \angle C = 180°$ (not possible)

41. (c)

42. (a) : Since, a_1, a_2, a_3 are in A.P.

$\Rightarrow \quad a_2 - a_1 = d = a_3 - a_2$

Now, $\tan^{-1}\left(\dfrac{d}{1+a_1a_2}\right)+\tan^{-1}\left(\dfrac{d}{1+a_2a_3}\right)$

$= \tan^{-1}\left(\dfrac{a_2-a_1}{1+a_1a_2}\right)+\tan^{-1}\left(\dfrac{a_3-a_2}{1+a_2a_3}\right)$

$= \tan^{-1}a_2 - \tan^{-1}a_1 + \tan^{-1}a_3 - \tan^{-1}a_2 = \tan^{-1}a_3 - \tan^{-1}a_1$

$= \tan^{-1}\left(\dfrac{a_3-a_1}{1+a_1a_3}\right) = \tan^{-1}\left(\dfrac{(a_3-a_2)+(a_2-a_1)}{1+a_1a_3}\right) = \tan^{-1}\left(\dfrac{2d}{1+a_1a_3}\right)$

43. (a)

44. (a) : We have, $\tan\left(\dfrac{A-B}{2}\right) = \sqrt{\dfrac{1-\cos(A-B)}{1+\cos(A-B)}}$

$= \sqrt{\dfrac{1-(31/32)}{1+(31/32)}} = \dfrac{1}{\sqrt{63}} \Rightarrow \dfrac{a-b}{a+b}\cot\dfrac{C}{2} = \dfrac{1}{\sqrt{63}}$

$\Rightarrow \dfrac{1}{9}\cot\dfrac{C}{2} = \dfrac{1}{\sqrt{63}} \Rightarrow \tan\dfrac{C}{2} = \dfrac{\sqrt{7}}{3}$

Now, $\cos C = \dfrac{1-\tan^2(C/2)}{1+\tan^2(C/2)} \Rightarrow \cos C = \dfrac{1-(7/9)}{1+(7/9)} = \dfrac{1}{8}$

$\therefore \quad c^2 = a^2 + b^2 - 2ab\cos C$

$\Rightarrow \quad c^2 = 25 + 16 - 40 \times \dfrac{1}{8} = 36 \Rightarrow c = 6$

45. (c) : Let $a = 6+\sqrt{12}, b = \sqrt{48}, c = \sqrt{24}$

Clearly, c is the smallest side. Therefore, the smallest angle C is given by

$\cos C = \dfrac{a^2+b^2-c^2}{2ab} = \dfrac{\sqrt{3}}{2} \Rightarrow \angle C = \dfrac{\pi}{6}$

46. (a) : Obviously, the angles are 30°, 45°, 105° .

$\therefore \quad a : b : c = \sin 30° : \sin 45° : \sin 105°$

$= \dfrac{1}{2} : \dfrac{1}{\sqrt{2}} : \dfrac{\sqrt{3}+1}{2\sqrt{2}} = \sqrt{2} : 2 : (\sqrt{3}+1)$

47. (b) : Let $a = 2, b = \sqrt{6}, c = \sqrt{3}+1$

$\therefore$ Smallest angle A is given by

$\therefore \quad \cos A = \dfrac{6+3+1+2\sqrt{3}-4}{2\sqrt{6}(\sqrt{3}+1)} \Rightarrow A = 45°$

48. (b)

49. (c) : $7\cos x + 5\sin x = 2\lambda + 1$

$|2\lambda+1| \le \sqrt{49+25} \Rightarrow |2\lambda+1| \le \sqrt{74}$

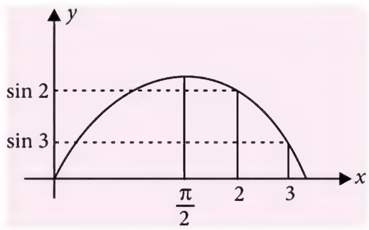
$-\sqrt{74} \le 2\lambda + 1 \le \sqrt{74}$

$\Rightarrow \quad -9.6 \le 2\lambda \le 7.6 \Rightarrow -4.8 \le \lambda \le 3.8$

$\therefore \quad \lambda = -4, -3, -2, -1, 0, 1, 2, 3$

$a\cos\theta + b\sin\theta = c$ has no solution if $|c| > \sqrt{a^2+b^2}$

50. (a) :



51. (d)

52. (b) : $a \le h_a = b\sin C$ and $b \le h_b = a\sin C$

$\Rightarrow \quad ab \le ab\sin^2 C \Rightarrow 1 \le \sin^2 C$

Hence, $\sin C = 1 \Rightarrow \angle C = \dfrac{\pi}{2}$

Hence, Statement-1 is true.

Clearly, Statement-2 is true but has no relevance with Statement-1.

53. (b)

54. (d) : $\dfrac{a^3+b^3+c^3+3abc}{2} > a^3$

$\Rightarrow \quad b^3 + c^3 + (-a)^3 - 3(-a)(b)(c) > 0$

$\Rightarrow \quad \dfrac{1}{2}(b+c-a)[(b-c)^2+(c+a)^2+(b+a)^2] > 0$

$\Rightarrow \quad b + c - a > 0 \Rightarrow b + c > a$

Similarly, $c + a > b$ and $a + b > c$

$\Rightarrow \quad a, b, c$ will form the sides of a triangle.

55. (a) : We have, $\cos A + \cos C = 2(1 - \cos B)$

$\Rightarrow \quad 2\cos\dfrac{A+C}{2}\cos\dfrac{A-C}{2} = 4\sin^2\dfrac{B}{2}$

$\Rightarrow \quad \dfrac{\cos\left(\dfrac{A-C}{2}\right)}{\sin\dfrac{B}{2}} = 2 \Rightarrow 2s - b = 2b \Rightarrow a + c = 2b$

Thus, a, b, c are in A.P.

Clearly, statement-2 is true.

56. (b) : $(\alpha - \beta) + \alpha + (\alpha + \beta) = \pi$ and $\alpha + \beta = \pi/2$

$\Rightarrow$ angles are $\pi/6, \pi/3, \pi/2 \Rightarrow a = R, b = \sqrt{3}R$, $c = 2R$

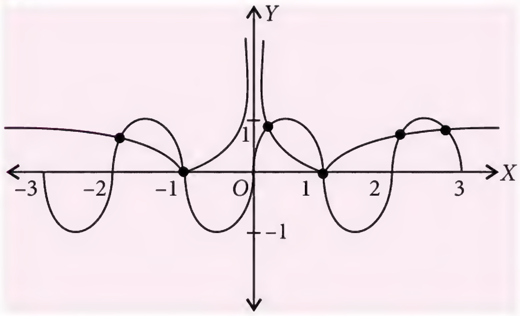
$\therefore \quad r = \dfrac{\Delta}{s} = \dfrac{\sqrt{3}}{3+\sqrt{3}}R \Rightarrow \dfrac{r}{R} = \dfrac{\sqrt{3}}{3+\sqrt{3}} = \dfrac{\sqrt{3}(3-\sqrt{3})}{6} = \dfrac{\sqrt{3}-1}{2}$

Also, $\dfrac{r}{2s} = \dfrac{\sqrt{3}R}{3+\sqrt{3}} \times \dfrac{1}{(3+\sqrt{3})R} = \dfrac{1}{(3+\sqrt{3})(\sqrt{3}+1)}$

$$\Rightarrow \quad \frac{r}{2s} = \frac{1}{6+4\sqrt{3}} = \frac{2-\sqrt{3}}{2\sqrt{3}} \Rightarrow \frac{r}{s} = \frac{2-\sqrt{3}}{\sqrt{3}}$$

57. (a) : Use the properties of modulus and greatest integer function.

58. (b) :



The graph shows 6 solutions.

59. (a) : $\sin^5 x - \cos^5 x = \dfrac{\sin x - \cos x}{\sin x \cos x}$

$$\Rightarrow \quad \sin x \cos x \left[\frac{\sin^5 x - \cos^5 x}{\sin x - \cos x}\right] = 1$$

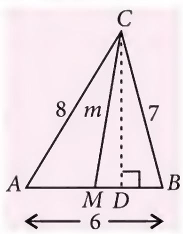
$\Rightarrow$ $\sin 2x[\sin^4 x + \sin^3 x\cos x + \sin^2 x\cos^2 x + \sin x\cos^3 x + \cos^4 x] = 2$

$\Rightarrow$ $(\sin 2x - 2)^2(\sin 2x + 2) = 0$

$\Rightarrow$ $\sin 2x = \pm 2$, which is not possible.

60. (d): Altitude from 'C' $= \dfrac{2\Delta}{c} = \dfrac{2}{6} \times \dfrac{21\sqrt{15}}{4} = \dfrac{7\sqrt{15}}{4}$

61. (c) : $2(9 + m^2) = 7^2 + 8^2$ where 'm' is the length of the median from vertex 'C'



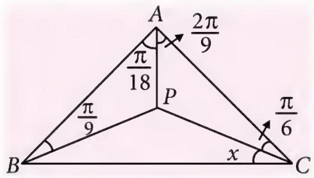
62. (b) : Length of the internal bisector of angle C'

$$= \frac{2\sqrt{abs(s-c)}}{a+b}$$

$$= \frac{2\sqrt{7 \cdot 8 \cdot \frac{21}{2}\left(\frac{21}{2} - 6\right)}}{7+8} = \frac{\sqrt{10584}}{15} = \frac{14\sqrt{6}}{5}$$

63. (a) : $\dfrac{PA}{\sin 20^\circ} = \dfrac{PB}{\sin 10^\circ} \Rightarrow \dfrac{PA}{PB} = \dfrac{\sin 20^\circ}{\sin 10^\circ}$

Similarly, $\dfrac{PB}{PC} = \dfrac{\sin x}{\sin(80^\circ - x)}, \dfrac{PC}{PA} = \dfrac{\sin 40^\circ}{\sin 30^\circ}$



Now, $\dfrac{PA}{PB} \times \dfrac{PB}{PC} \times \dfrac{PC}{PA} = 1$

$$\Rightarrow \quad \frac{\sin 20^\circ}{\sin 10^\circ} \times \frac{\sin x}{\sin(80^\circ - x)} \times \frac{\sin 40^\circ}{\sin 30^\circ} = 1$$

$\Rightarrow$ $\sin(80^\circ - x) - \sin x = 2\sin 50^\circ \sin x$

$\Rightarrow$ $2\sin(40^\circ - x)\cos 40^\circ = 2\sin 50^\circ \sin x$

$\Rightarrow$ $\sin(40^\circ - x) = \sin x \Rightarrow x = 20^\circ$

64. (b) : $\angle A = \angle C = 50^\circ$

65. (c) : $\angle ABC = 80^\circ \Rightarrow AC$ is longest side

66. (d)

67. (c) : $\displaystyle\lim_{n\to\infty} \sum_{r=1}^{n} \cot^{-1}\left(2^{r+1} + \frac{1}{2^r}\right)$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} \tan^{-1}\left(\frac{2^r}{1 + 2^r \cdot 2^{r+1}}\right)$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} \tan^{-1}\left(\frac{2^{r+1} - 2^r}{1 + 2^r \cdot 2^{r+1}}\right)$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} (\tan^{-1} 2^{r+1} - \tan^{-1} 2^r)$$

$$= \lim_{n\to\infty} [\tan^{-1} 2^{n+1} - \tan^{-1} 2]$$

$$= \tan^{-1}\infty - \tan^{-1}2 = \frac{\pi}{2} - \tan^{-1} 2 = \cot^{-1} 2$$

68. (d)

69. (A) → (q), (B) → (q), (C) → (s), (D) → (r)

$$r_1r_2 + r_2r_3 + r_3r_1 = \frac{r_1r_2r_3}{r} = s^2 = 144$$

$\Rightarrow$ $4(r_2 + r_3) + r_2r_3 = 2r_2r_3 = 144 \Rightarrow r_2r_3 = 72,$

$\Rightarrow$ $r_2 + r_3 = 18 \Rightarrow |r_2 - r_3| = 6 \Rightarrow r_2 = 6, r_3 = 12$

Now, $\dfrac{r}{r_1} = \dfrac{s-a}{s} = \dfrac{1}{2} \Rightarrow a = 6$

Similarly, $b = 8$, $c = 10 \Rightarrow ABC$ is right angle triangle.

Smallest angle is $\sin^{-1}\dfrac{3}{5}, \Delta = \dfrac{1}{2} \times 6 \times 8 = 24$ sq. units

$R = 5$

70. (A) → (q), (B) → (p, s), (C) → (r)

(A) $\lambda=-\frac{\pi}{6}+\frac{\pi}{3}=\frac{\pi}{6}$ and $\mu=\frac{2\pi}{3}$

$\therefore\ \lambda+\mu=\frac{\pi}{6}+\frac{2\pi}{3}=\frac{5\pi}{6}$ and $\mu-\lambda=\frac{2\pi}{3}-\frac{\pi}{6}=\frac{\pi}{2}$

(B) $\lambda=\pi-\frac{7\pi}{6}=-\frac{\pi}{6}$ and $\mu=\cos^{-1}\left(-\sin\frac{5\pi}{6}\right)$

$=\cos^{-1}\left(-\frac{1}{2}\right)=\cos^{-1}\left(\cos\frac{2\pi}{3}\right)=\frac{2\pi}{3}$

$\therefore\ \lambda+\mu=-\frac{\pi}{6}+\frac{2\pi}{3}=\frac{\pi}{2},\ \mu-\lambda=\frac{2\pi}{3}+\frac{\pi}{6}=\frac{5\pi}{6}$

(C) $\lambda=\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)=\sin^{-1}\left(\sin\left(-\frac{\pi}{3}\right)\right)=-\frac{\pi}{3}$

$\mu=\sin^{-1}\left(\cos\left(\sin^{-1}\frac{\sqrt{3}}{2}\right)\right)=\sin^{-1}\left(\cos\frac{\pi}{3}\right)$

$=\sin^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{6}$

$\therefore\ \lambda+\mu=-\frac{\pi}{6}$ and $\mu-\lambda=\frac{\pi}{2}$

71. (3) : $\theta=\cot^{-1}7+\cot^{-1}8+\cot^{-1}18$

$=\tan^{-1}\left(\frac{1}{7}\right)+\tan^{-1}\left(\frac{1}{8}\right)+\tan^{-1}\left(\frac{1}{18}\right)$

$=\tan^{-1}\left(\dfrac{\frac{1}{7}+\frac{1}{8}+\frac{1}{18}-\frac{1}{7}\cdot\frac{1}{8}\cdot\frac{1}{18}}{1-\frac{1}{7}\cdot\frac{1}{8}-\frac{1}{8}\cdot\frac{1}{18}-\frac{1}{18}\cdot\frac{1}{7}}\right)$

$=\tan^{-1}\left(\frac{1}{3}\right)=\cot^{-1}3$

$\therefore$ $\cot\theta=3$

72. (6) : $2\sin\theta=(r^2-1)^2+2$

This is possible when $\sin\theta=1$, $r^2=1$, $r=\pm1$

$\therefore\ \theta=\frac{\pi}{2},\frac{5\pi}{2},\frac{9\pi}{2}$

$\therefore$ Number of values of the pair(r,θ) is 6.

73. (4)

74. (1) : $\because\ \frac{\pi}{2}-\cos^{-1}x+\frac{\pi}{2}-\cos^{-1}y=\pi$

$\Rightarrow$ $\cos^{-1}x+\cos^{-1}y=0$

$\Rightarrow\ \cos^{-1}\{xy-\sqrt{(1-x^2)}\sqrt{(1-y^2)}\}=0$

$\Rightarrow\ xy-\sqrt{(1-x^2)(1-y^2)}=1$

$\Rightarrow$ $(xy-1)^2=(1-x^2)(1-y^2)$

$\Rightarrow$ $x^2y^2+1-2xy=1-x^2-y^2+x^2y^2$

$\Rightarrow$ $x^2+y^2-2xy=0\Rightarrow(x-y)^2=0\Rightarrow x=y$

$\therefore$ $\lambda=1$

75. (0) : $\log_{|\sin x|}(1+\cos x)=2\Rightarrow 1+\cos x=|\sin x|^2$

$\Rightarrow$ $1+\cos x=1-\cos^2x\Rightarrow\cos x(1+\cos x)=0$

But $(1+\cos x)\neq0\Rightarrow\cos x=0\Rightarrow\sin x=1$.

But $\sin x=1$ is not possible because the base of log can not be 1. Hence no solution.

76. (4) : $S=\sum_{r=1}^{50}\tan^{-1}\left(\dfrac{(1+r+r^2)-(1-r+r^2)}{1+(1+r+r^2)(1-r+r^2)}\right)$

$=\sum_{r=1}^{50}[\tan^{-1}(1+r+r^2)-\tan^{-1}(1-r+r^2)]$

= {at $r=50$ value of $\tan^{-1}(1+r+r^2)$}
− {at $r=1$ value of $\tan^{-1}(1-r+r^2)$}

$=\tan^{-1}(1+50+50^2)-\tan^{-1}1$

$=\tan^{-1}\left(\dfrac{1+50+50^2-1}{1+1+50+50^2}\right)=\tan^{-1}\left(\dfrac{2550}{2552}\right)$

$\therefore\ \tan S=\frac{2550}{2552}\ \ \therefore\ \frac{2550}{638}\cot S=4$

77. (6) : $\cos C=\dfrac{1-\frac{7}{9}}{1+\frac{7}{9}}=\frac{1}{8}$

$\therefore\ c^2=25+16-2\cdot5\cdot4\cdot\frac{1}{8}=36\ \Rightarrow\ c=6$

78. (4) : $a^2=b^2+c^2-2bc\cos A$

$4+c^2-\frac{\sqrt{3}}{2}\cdot4c-5=0\ \Rightarrow\ c^2-2\sqrt{3}\cdot c-1=0$

Now, $c_1+c_2=2\sqrt{3};\ c_1c_2=-1$

$(c_1-c_2)^2=(c_1+c_2)^2-4c_1c_2=(2\sqrt{3})^2-4(-1)=16$

79. (2) : $3a=b+c$

$\Rightarrow$ $4a=2s\Rightarrow s=2a$

$\therefore\ \cot\frac{B}{2}\cot\frac{C}{2}=\frac{s}{s-a}=\frac{2a}{2a-a}=2$

80. (8) : $r_1-r=4R\sin^2\frac{A}{2}\ \Rightarrow\ 4=20\sin^2\frac{A}{2}$

$\Rightarrow\ \sin\frac{A}{2}=\frac{1}{\sqrt{5}}\ \ \tan A=\dfrac{2\times\frac{1}{2}}{1-\frac{1}{4}}=\frac{4}{3}\ \Rightarrow\ 6\tan A=8$

BRAIN@WORK

PROBABILITY

1. Three ordinary and fair dice are rolled simultaneously. The probability of the sum of outcomes being atleast equal to 8, is equal to
(a) 81/108 (b) 27/216
(c) 81/216 (d) 181/216

2. A person takes a step forward with probability p and takes a step backward with probability q, where $p + q = 1$. The probability that after $(2n + 1)$ steps the person is only one step away from his initial position, is equal to
(a) ${}^{2n+1}C_n \cdot p^{n+1} \cdot q^n$ (b) ${}^{2n+1}C_n \cdot p^n \cdot q^{n+1}$
(c) ${}^{2n+1}C_n \cdot p^n \cdot q^n$ (d) $2 \cdot {}^{2n+1}C_n \cdot p^n\, q^{n+1}$

3. Three numbers are selected simultaneously from the set $\{1, 2, 3, ..., 25\}$. The probability that the product of selected numbers is divisible by 4, is equal to
(a) 1/115 (b) 98/115
(c) 773/1150 (d) 963/1150

4. Two numbers n_1 and n_2 are chosen at random (without replacement) from the set $\{1, 2, 3, ..., 5n\}$, the probability that $n_1^4 - n_2^4$ is divisible by 5, is equal to
(a) $\frac{n-1}{5n-1}$ (b) $\frac{4(4n-1)}{5(5n-1)}$
(c) $\frac{17n-5}{5(5n-1)}$ (d) $\frac{8n}{5(5n-1)}$

5. 'X' follows a binomial distribution with parameters 'n' and 'p'. 'Y' follows a binomial distribution with parameters 'm' and 'p'. If X and Y are independent then $P(X = r|_{X+Y=r+s})$ is equal to
(a) $\frac{{}^nC_r \cdot {}^mC_s}{{}^{m+n}C_r}$ (b) $\frac{{}^nC_r \cdot {}^mC_s}{{}^{m+n}C_{r+s}}$
(c) $\frac{{}^mC_s \cdot {}^nC_r}{{}^{m+n}C_{s-1}}$ (d) $\frac{{}^mC_s \cdot {}^nC_r}{{}^{m+n}C_{r+s-1}}$

6. If two events A and B are such that $P(\bar{A}) = \frac{3}{10}$, $P(B) = \frac{2}{5}$ and $P(A \cap \bar{B}) = \frac{1}{2}$, then $P(B\,|\,A \cup \bar{B})$ is equal to
(a) 1/2 (b) 1/3 (c) 1/4 (d) 2/3

7. $3m$ fair dice are each rolled $2n$ times. The probability that the scores 1, 2, 3, 4, 5, 6 each appear mn times, is equal to
(a) $\frac{(6mn)!}{6!((mn)!)^6}\cdot\left(\frac{1}{6}\right)^{mn}$ (b) $\frac{(6mn)!}{(6!(mn)!)^6}\cdot\left(\frac{1}{6}\right)^{6mn}$
(c) $\frac{(6mn)!}{((mn)!)^6}\cdot\left(\frac{1}{6}\right)^{mn}$ (d) $\frac{(6mn)!}{((mn)!)^6}\cdot\left(\frac{1}{6}\right)^{6mn}$

8. Two persons are selected randomly from n persons seated in a row $(n \geq 3)$. The probability that the selected persons are not seated consecutively, is equal to
(a) $\frac{n-2}{n}$ (b) $\frac{n-1}{n}$ (c) $\frac{n+2}{n+3}$ (d) $\frac{n-2}{n-1}$

9. A real estate man has eight master keys to open several new homes. Only one master key will open any given home. If 40% of these homes are usually left unlocked, the probability that the real estate man can get into a specific home, if it is given that he selected 3 keys randomly before leaving his office, is equal to
(a) 5/8 (b) 3/8 (c) 3/4 (d) 1/4

10. A man alternatively tosses a fair coin and rolls a fair ordinary dice. He starts with the coin. The probability that he gets a tail on the coin before getting 5 or 6 on the dice, is equal to
(a) 3/4 (b) 1/2 (c) 1/3 (d) 2/3

11. For three events E_1, E_2 and E_3. P(exactly one of the events E_1 or E_2 occur) = P(exactly one of the event E_2 or E_3 occur) = P(exactly one of the events E_3 or E_1 occur)

$= p$ and P(all the three events occur simultaneously) $= p^2$, where $p \in \left(0, \frac{1}{2}\right)$. The probability that atleast one of these events occur is

(a) $\frac{3p+2p^2}{2}$ (b) $\frac{p+3p^2}{2}$

(c) $\frac{3p+p^2}{2}$ (d) $\frac{3p+2p^2}{4}$

12. A bag contains 'W' white balls and 'R' red balls. Two players P_1 and P_2 alternatively draw a ball from the bag, replacing the ball each time after the draw, till one of them draws a white ball and wins the game. 'P_1' begins the game. The probability of P_2 being the winner, is equal to

(a) $\frac{W^2}{(W+2R)R}$ (b) $\frac{R}{(2W+R)}$

(c) $\frac{R^2}{(2W+R)W}$ (d) $\frac{R}{(W+2R)}$

13. A bag contains 4 tickets numbered 1, 2, 3, 4 and another bag contains 6 tickets numbered 2, 4, 6, 7, 8, 9. One bag is chosen and a ticket is drawn. The probability that the ticket bears the number 4, is equal to

(a) 5/12 (b) 5/24 (c) 7/12 (d) 19/24

14. A number of the form $7^{n_1} + 7^{n_2}$ is formed, by selecting the numbers n_1 and n_2 from the set {1, 2, 3, ..., 99, 100} with replacement. The probability of the formed number being divisible by 5, is equal to

(a) 1/8 (b) 1/4 (c) 1/16 (d) 1/2

15. 'P_1' is the probability that a statement by A is true and 'P_2' has the similar meaning for B. A and B agree in making a statement S. The probability that statement is correct is

(a) $\frac{P_1P_2}{P_1P_2 + P_1(1-P_2)(1-P_1)}$

(b) $\frac{P_1P_2}{P_1P_2 + P_2(1-P_1)(1-P_2)}$

(c) $\frac{P_1P_2}{P_1+P_2+1}$ (d) $\frac{P_1P_2}{1-P_1-P_2+2P_1P_2}$

16. A bag contains four tickets marked with numbers 112, 121, 211, 222. One ticket is drawn at random from the bag. Let $E_i (i = 1, 2, 3)$ denote the event that i^{th} digit on the selected ticket is 2, then which of the following is not correct?

(a) E_1 and E_2 are independent

(b) E_2 and E_3 are independent

(c) E_3 and E_1 are independent

(d) E_1, E_2, E_3 are independent

17. A bag B_1 has 3 white balls and 2 red balls. Another bag B_2 has 4 white and 6 red balls. A ball is drawn randomly from bag B_1 and without seeing it's colour, is being put in bag B_2. Now a ball is drawn from bag B_2. The probability of both the drawn balls, of being same colour, is

(a) 41/55 (b) 31/55

(c) 29/55 (d) none of these

18. A natural number 'n' is selected at random from the set of first 100 natural numbers. The probability that $n + \frac{100}{n} \le 50$ is equal to

(a) 9/10 (b) 39/50

(c) 9/20 (d) none of these

19. For two events A and B, $P(A) = P(A \mid B) = \frac{1}{4}$ and $P(B \mid A) = \frac{1}{2}$, then which of the following is not correct?

(a) $P(A' \mid B) = \frac{3}{4}$ (b) $P(B' \mid A') = \frac{1}{2}$

(c) $P(A \cup B) = \frac{3}{4}$ (d) $P(A \cap B) = \frac{1}{8}$

20. If the probability of choosing an integer 'n' out of $2m$ integers {1, 2, 3, ..., $2m - 1$, $2m$} is inversely proportional to n^4 ($1 \le n \le 2m$), then the probability of the chosen number being odd, is

(a) $\frac{1}{2}$ (b) $< \frac{1}{2}$

(c) $> \frac{1}{2}$ (d) none of these

21. In a bag there are 15 red and 5 white balls. Two balls are drawn in succession, without replacement. The first drawn ball is found to be red. The probability that second ball is also red, is equal to

(a) 3/10 (b) 7/10 (c) 5/19 (d) 14/19

22. Two squares are chosen from squares of a ordinary chess board. It is given that the selected squares do not belong to the same row or column. The probability that the chosen squares are of same colour, is equal to

(a) 25/49 (b) 32/49 (c) 25/64 (d) 1/2

23. Two fair dice are rolled simultaneously. One of the dice shows four. The probability of other dice showing six, is equal to

(a) 2/11 (b) 1/18 (c) 1/6 (d) 1/36

24. Four integers are selected randomly and are multiplied. The probability of this product being divisible by 5 but not by 10, is equal to

(a) $\frac{175}{10^4}$ (b) $\frac{369}{10^4}$ (c) $\frac{3471}{10^4}$ (d) $\frac{1}{32}$

25. Two events A and B are such that $P(A) > 0$ and $P(B) \neq 1$. The expression $P(\bar{A} \mid \bar{B})$ is also equal to

(a) $1 - P(A \mid B)$ (b) $1 - P(\bar{A} \mid B)$

(c) $\frac{1 - P(A \cup B)}{P(\bar{B})}$ (d) $\frac{1 - P(A \cap B)}{P(\bar{B})}$

26. Consider the set of integers {10, 11, 12, ..., 98, 99}. By seeing the number a person A will laugh if the product of the digits of the number is 12. He chooses three numbers one at a time from this set of integers, randomly and with replacement. The probability that he will laugh atleast once, is equal to

(a) $1-\left(\frac{43}{45}\right)^3$ (b) $1-\left(\frac{3}{5}\right)^3$

(c) $1-\left(\frac{4}{25}\right)^3$ (d) none of these

27. The sum of two natural numbers n_1 and n_2 is known to be equal to 100. The probability that their product being greater than 1600, is equal to

(a) 20/33 (b) 58/99 (c) 13/33 (d) 59/99

28. Let 'head' means one and 'tail' means two and the coefficients of the equation $ax^2 + bx + c = 0$ are chosen by tossing three fair coins. The probability that the roots of the equation are non-real, is equal to

(a) 5/8 (b) 7/8 (c) 3/8 (d) 1/8

29. A committee consists of 9 experts taken from three institutions A, B and C, of which 2 are from A, 3 from B and 4 from C. If three experts resign from the committee, then the probability of exactly two of the resigned experts being from the same institution, is equal to

(a) 4/7 (b) 25/84 (c) 55/84 (d) 37/84

30. A person while dialing a telephone number, forgets the last three digits of the number but remembers that exactly two of them are same. He dials the number randomly. The probability that he dialed the correct number, is equal to

(a) 1/135 (b) 1/27 (c) 1/54 (d) 1/270

31. Two subsets A and B of a set S consisting of 'n' elements are constructed randomly. The probability that $A \cap B = \phi$ and $A \cup B = S$ is equal to

(a) $1-\left(\frac{3}{4}\right)^n$ (b) $\left(\frac{3}{4}\right)^n$

(c) $\frac{1}{2^n}$ (d) $\frac{1}{3^n}$

32. Consider all functions that can be defined from the set $A = \{1, 2, 3\}$ to the set $B = \{1, 2, 3, 4, 5\}$. A function $f(x)$ is selected at random from these functions. The probability that, selected function satisfies $f(i) \leq f(j)$ for $i < j$, is equal to

(a) 6/25 (b) 7/25 (c) 2/5 (d) 12/25

33. A fair coin is tossed 'n' number of times. The probability that head will turn up an even number of times, is equal to

(a) $\frac{n-1}{2n}$ (b) $\frac{1}{2}$

(c) $\frac{n+1}{2n}$ (d) $\frac{2^{n-1}-1}{2^n}$

34. Two numbers n_1 and n_2 are selected from the set {1, 2, 3, ..., n} without replacement. The value of $P(n_1 \leq r \mid n_2 \leq r)$, where $r \leq n$ is equal to

(a) $\frac{r}{n(n-1)}$ (b) $\frac{r}{n}$

(c) $\frac{r-1}{n-1}$ (d) $\frac{r(r-1)}{n(n-1)}$

35. A and B play a game where each is asked to select a number from 1 to 25. If the numbers selected by A and B match, both of them win a prize. The probability that they win their third prize on 5th game is equal to

(a) $\frac{6 \cdot (24)^2}{(25)^5}$ (b) $\frac{6 \cdot (21)^2}{(25)^5}$

(c) $\frac{(24)^2}{(25)^2}$ (d) $\frac{(21)^2}{(25)^5}$

36. The probability that an archer hits the target when it is windy is equal to 2/5, when it is not windy his probability of hitting the target is 7/10. On any shot the probability of gust of wind is $\frac{3}{10}$. The probability that there is no gust of wind on the occasion when he missed the target, is equal to

(a) 5/13 (b) 18/39 (c) 7/13 (d) 23/39

37. A and B each throw a fair dice. The probability that A's throw is not greater than B's throw, is equal to

(a) 1/3 (b) 2/3 (c) 7/12 (d) 5/12

38. A fair dice is rolled three times. Let E_1 be the event that even number comes up on the first roll, E_2 be the event that even number appears on second and third roll and E_3 be the event that all the roll result in same number. Which of the following statements is not correct?

(a) E_1 and E_2 are not independent
(b) E_2 and E_3 are independent
(c) E_1 and E_3 are independent
(d) None of these

39. A and B are two events such that $P(A)=\frac{1}{2}$, $P(B)=\frac{2}{3}$. Then which of the following is not correct?

(a) $P(A\cup B)\geq\frac{2}{3}$ (b) $P(A\cap B')\leq\frac{1}{3}$
(c) $0\leq P(A\cap B)\leq\frac{1}{2}$ (d) None of these

40. Three numbers are selected from the set {1, 2, 3, ..., 23, 24}, without replacement. The probability that the formed numbers form an A.P. is equal to

(a) 11/23 (b) 12/23
(c) 3/46 (d) None of these

SOLUTIONS

1. (d) : Let the outcome on dice be n_1, n_2 and n_3, we first try to find the number of ways in which sum of outcomes is at the most equal to seven.
For this, $n_1 + n_2 + n_3 \leq 7$, where $n_i \in [1, 6]$
$\Rightarrow$ $(n_1 - 1) + (n_2 - 1) + (n_3 - 1) + y_4 = 4$
i.e., $y_1 + y_2 + y_3 + y_4 = 4$,
where y_1, y_2, y_3, y_4 are non-negative integers.
This can happen in ${}^{4+4-1}C_4$ *i.e.,* ${}^7C_4$ ways.

Thus, required probability $=1-\frac{{}^7C_4}{6^3}=\frac{181}{216}$

2. (c) : Clearly, the person should take $(n + 1)$ steps in one direction and n steps in the another direction.
Thus, required probability
$= {}^{2n+1}C_n\, p^n \cdot q^{n+1} + {}^{2n+1}C_n q^n \cdot p^{n+1}$
$= {}^{2n+1}C_n\, p^n \cdot q^n(p + q) = {}^{2n+1}C_n p^n \cdot q^n$

3. (c) : Product can't be divisible by 4 in the following mutually exclusive cases:

(i) All the selected numbers are odd :
Corresponding number of ways $= {}^{13}C_3 = 286$
(ii) Two of the selected numbers are odd and another is a multiple of 2 only.

Corresponding number of ways $= {}^{13}C_2 \cdot {}^6C_1 = 468$

Thus, required probability $=1-\frac{(286+468)}{{}^{25}C_3}=\frac{773}{1150}$

4. (c) : $n_1^4 - n_2^4 = (n_1^2 + n_2^2)(n_1 + n_2)(n_1 - n_2)$
We can segregate the numbers as multiple of five 5λ, $5\lambda + 1$, $5\lambda + 2$, $5\lambda + 3$, $5\lambda + 4$, as follows :

$R_1 \rightarrow 1$ 6 ....$5n - 4 \rightarrow 5\lambda + 1$
$R_2 \rightarrow 2$ 7 ....$5n - 3 \rightarrow 5\lambda + 2$
$R_3 \rightarrow 3$ 8 ....$5n - 2 \rightarrow 5\lambda + 3$
$R_4 \rightarrow 4$ 9 ....$5n - 1 \rightarrow 5\lambda + 4$
$R_5 \rightarrow 5$ 10 ....$5n \rightarrow 5\lambda$

We can select either both the number from R_5, or any two numbers from the first four rows. (As square of any number that is not a multiple of three will be in the form of $5\lambda' + 1$ or $5\lambda' - 1$).
Thus, required probability

$$=\frac{{}^nC_2+{}^{4n}C_2}{{}^{5n}C_2}=\frac{n(n-1)+4n(4n-1)}{5n(5n-1)}=\frac{17n-5}{5(5n-1)}$$

5. (b) : $P(X = r|_{X+Y=r+s})$

$$=\frac{P(X=r\cap X+Y=r+s)}{P(X+Y=r+s)}=\frac{P(X=r)\cdot P(Y=s)}{P(X+Y=r+s)}$$

Now, $P(X = r) = {}^nC_r\, p^r \,.\, (1 - p)^{n-r}$,
$P(Y = s) = {}^mC_s\, p^s \,.\, (1 - p)^{m-s}$,

and, $P(X+Y=r+s)=\sum_{i=0}^{n}P(X=i)$

$P(Y = r + s - i) = \sum_{i=0}^{n}{}^nC_i p^i(1-p)^{n-i}\,.\,{}^mC_{r+s-i}$
$\cdot\, p^{r+s-i} \,.\, (1 - p)^{m-r-s+i}$

$=p^{r+s}\cdot(1-p)^{m+n-r-s}\sum_{i=0}^{n}{}^nC_i\,.\,{}^mC_{r+s-i}$

$= p^{r+s} \,.\, (1 - p)^{m+n-r-s} \,.\, {}^{m+n}C_{r+s}$

Thus, required probability $=\frac{{}^nC_r\cdot{}^mC_s}{{}^{m+n}C_{r+s}}$

6. (c) : $P(B\,|\,A\cup\bar{B})=\frac{P(B\cap(A\cup\bar{B}))}{P(A\cup\bar{B})}=\frac{P(A\cap B)}{P(A\cup\bar{B})}$

Now, $P(A\cap\bar{B})=P(A)-P(A\cap B)$

$\Rightarrow$ $P(A\cap B)=P(A)-P(A\cap\bar{B})=\frac{7}{10}-\frac{1}{2}=\frac{1}{5}$

Also, $P(A\cup\bar{B})=P(A)+P(\bar{B})-P(A\cap\bar{B})$

$=\frac{7}{10}+\frac{3}{5}-\frac{1}{2}=\frac{4}{5}$ $\Rightarrow$ $P(B\,|\,A\cup\bar{B})=\frac{1}{4}$

7. (d) : Total number of ways of arranging $6mn$ objects to equal groups, consisting of mn objects in each groups are numbered as 1, 2, ..., 6 $=\frac{(6mn)!}{(mn!)^6}$

Corresponding probability for each of these ways

$$=\left(\frac{1}{6}\right)^{mn}\cdot\left(\frac{1}{6}\right)^{mn}\ldots\ldots\left(\frac{1}{6}\right)^{mn}=\left(\frac{1}{6}\right)^{6mn}$$

Thus, required probability $=\dfrac{(6mn)!}{(mn!)^6}\cdot\left(\dfrac{1}{6}\right)^{6mn}$

8. (a) : Total ways of selecting two persons = ${}^nC_2$

$$=\frac{n(n-1)}{2}$$

Total ways of selecting two consecutively seated persons $= {}^{n-1}C_1 = (n-1)$

Thus, required probability $=1-\dfrac{(n-1)2}{n(n-1)}=\dfrac{n-2}{n}$

9. (a) : E_1 : Specific home is locked.
E_2 : Specific home is unlocked.
A : Real estate man get into the home.

$$P(E_2)=\frac{40}{100}=\frac{2}{5},\ P(E_1)=\frac{3}{5}$$

$P(A) = P(E_1)\cdot P(A/E_1) + P(E_2)\cdot P(A/E_2)$

$$=\frac{3}{5}\cdot\frac{3}{8}+\frac{2}{5}\cdot 1=\frac{25}{40}=\frac{5}{8}$$

10. (a) : Probability of getting 5 or 6 on a specific roll of dice $=\dfrac{2}{6}=\dfrac{1}{3}$

and, probability of getting a tail $=\dfrac{1}{2}$

The desired outcome can happen, in general, on $(2r+1)^{\text{th}}$ trial. That means first $2r$ trials should result neither in tail nor in 5 or 6 and $(2r+1)^{\text{th}}$ trial must result in tail.

If the corresponding probability is p, then

$$p_r=\left(\frac{1}{2}\right)^r\cdot\left(\frac{2}{3}\right)^r\cdot\frac{1}{2}=\frac{1}{2}\cdot\left(\frac{1}{3}\right)^r$$

Thus, required probability

$$=\sum_{r=0}^{\infty}p_r=\frac{1}{2}\sum_{r=0}^{\infty}\left(\frac{1}{3}\right)^r=\frac{3}{4}$$

11. (a) : Required probability

$= \Sigma P(E_i) - \Sigma P(E_i \cap E_j) + P(E_1 \cap E_2 \cap E_3)$

We have, $P(E_1) + P(E_2) - 2P(E_1 \cap E_2) = p$

$P(E_2) + P(E_3) - 2P(E_2 \cap E_3) = p$

$P(E_1) + P(E_3) - 2P(E_1 \cap E_3) = p$

$$\Rightarrow\ \Sigma P(E_i)-\Sigma P(E_i\cap E_j)=\frac{3p}{2}$$

Thus, required probability $=\dfrac{3p}{2}+p^2=\dfrac{3p+2p^2}{2}$

12. (d) : Probability of drawing a white ball at any draw

$$=\frac{W}{W+R}$$

Now, P_2 can be the winner in general on $2r^{\text{th}}$, $r \geq 1$, draw. That means in the first $(2r-1)$ draws no player draws a white ball and $2r^{\text{th}}$ draw results in a white ball for P_2.

If the corresponding probability is p_r, then

$$p_r=\left(\frac{R}{W+R}\right)^r\cdot\left(\frac{R}{W+R}\right)^{r-1}\cdot\left(\frac{W}{W+R}\right)=\left(\frac{R}{W+R}\right)^{2r}\cdot\frac{W}{R}$$

Hence, required probability $=\displaystyle\sum_{r=1}^{\infty}p_r$

$$=\frac{W}{R}\sum_{r=1}^{\infty}\left(\frac{R}{W+R}\right)^{2r}=\frac{R}{(W+2R)}$$

13. (b) : E_1 : First bag is chosen, $P(E_1)=\dfrac{1}{2}$

E_2 : Second bag is chosen, $P(E_2)=\dfrac{1}{2}$

A : Drawn number is 4.

Now, $P(A) = P(E_1)\cdot P(A/E_1) + P(E_2)\cdot P(A/E_2)$

$$=\frac{1}{2}\cdot\frac{1}{4}+\frac{1}{2}\cdot\frac{1}{6}=\frac{5}{24}$$

14. (a) : We know that

$7^{4\lambda}$ ends with 1,

$7^{4\lambda+1}$ ends with 7,

$7^{4\lambda+2}$ ends with 9,

and $7^{4\lambda+3}$ ends with 3.

Now, $7^{\lambda_1}+7^{\lambda_2}$ will be divisible by 5, if it ends with either 0 or 5. But it can never end with 5 and can end with zero in the two cases :

(i) $n_1 = 4\lambda$, $n_2 = 4\lambda + 2$

(ii) $n_1 = 4\lambda + 1$, $n_2 = 4\lambda + 3$

Thus, total favourable ways of selecting n_1 and n_2.

$= {}^{25}C_1\cdot{}^{25}C_1 + {}^{25}C_1\cdot{}^{25}C_1 = 2\cdot 25\cdot 25$

Thus, required probability $=\dfrac{2\cdot 25\cdot 25}{100\cdot 100}=\dfrac{1}{8}$

15. (d) : Let E_1 : Statement is true

E_2 : Statement is false

M : A and B agree

$P(E_1) = P(E_2) = \dfrac{1}{2}$

$P(M) = P(E_1)\cdot P(M/E_1) + P(E_2)\cdot P(M/E_2)$

$$=\frac{1}{2}\cdot P_1P_2+\frac{1}{2}\cdot(1-P_1)(1-P_2)$$

$$=\frac{P_1P_2+(1-P_1)(1-P_2)}{2}$$

$$\text{Now, } P(E_1/M) = \frac{P(A \cap E_1)}{P(M)} = \frac{P(E_1)\cdot P(A/E_1)}{P(M)}$$

$$= \frac{P_1P_2}{P_1P_2 + (1-P_1)(1-P_2)} = \frac{P_1P_2}{1-P_1-P_2+2P_1P_2}$$

16. (d) : $P(E_1) = \frac{2}{4} = \frac{1}{2}, P(E_2) = \frac{2}{4} = \frac{1}{2}, P(E_3) = \frac{2}{4} = \frac{1}{2}$

$$P(E_1 \cap E_2) = \frac{1}{4} = P(E_1)\cdot P(E_2)$$

$\Rightarrow$ 'E_1' and 'E_2' are independent.

$$P(E_1 \cap E_3) = \frac{1}{4} = P(E_1)\cdot P(E_3)$$

$\Rightarrow$ 'E_1' and 'E_3' are independent.

$$P(E_2 \cap E_3) = \frac{1}{4} = P(E_2)\cdot P(E_3)$$

$\Rightarrow$ 'E_2' and 'E_3' are independent.

$$P(E_1 \cap E_2 \cap E_3) = \frac{1}{4} \neq P(E_1)\cdot P(E_2)\cdot P(E_3)$$

$\Rightarrow$ E_1, E_2 and E_3 are not independent.

17. (c) : E_1 : Ball drawn from B_1 is white.
E_2 : Ball drawn from B_1 is red.
A : Both the drawn balls are of same colour.
$P(A) = P(E_1) \cdot P(A/E_1) + P(E_2) \cdot P(A/E_2)$

$$= \frac{3}{5}\cdot\frac{5}{11} + \frac{2}{5}\cdot\frac{7}{11} = \frac{29}{55}$$

18. (c) : $n + \frac{100}{n} \le 50 \Rightarrow n^2 - 50n + 100 \le 0$

$\Rightarrow$ $25 - 5\sqrt{21} \le n \le 25 + 5\sqrt{21}$

$\Rightarrow$ $n = 3, 4, 5, \ldots\ldots\ldots, 47$

Thus, favourable number of ways = 45.

Thus, required probability $= \frac{45}{100} = \frac{9}{20}$

19. (c) : $P(A) = P(A/B) = \frac{1}{4}$

$\Rightarrow$ A and B are independent.

Hence, $P(B/A) = P(B) = \frac{1}{2}$

Now, A', B; A', B' will also independent.

Thus, $P(A'/B) = P(A') = 1 - P(A) = \frac{3}{4}$

$$P(B'/A') = P(B') = 1 - P(B) = \frac{1}{2}$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{1}{4} + \frac{1}{2} - \frac{1}{4}\cdot\frac{1}{2} = \frac{5}{8}$$

$$P(A \cap B) = P(A)\cdot P(B) = \frac{1}{8}$$

20. (c) : Let's denote the probability of choosing the integer n by p_n, then

$$p_n = \frac{\lambda}{n^4}, n \in [1, 2m]$$

Now, $\sum_{n=1}^{2m} p_n = \lambda \sum_{n=1}^{2m} \frac{1}{n^4} = 1 \Rightarrow \lambda = \frac{1}{\sum_{n=1}^{2m} \frac{1}{n^4}}$

Now, required probability

$$= \sum_{n=1}^{m} p_{2n-1} = \lambda . \sum_{n=1}^{m} \frac{1}{(2n-1)^4} = \frac{\sum_{n=1}^{m} \frac{1}{(2n-1)^4}}{\sum_{n=1}^{2m} \frac{1}{n^4}}$$

Now, $\sum_{n=1}^{2m} \frac{1}{n^4} = \sum_{n=1}^{m} \frac{1}{(2n)^4} + \sum_{n=1}^{m} \frac{1}{(2n-1)^4} < 2\sum_{n=1}^{m} \frac{1}{(2n-1)^4}$

Thus, required probability > $\frac{1}{2}$.

21. (d) : Total number of ways of selecting the second ball = ${}^{19}C_1$ = 19.
Total number of ways of selecting the second ball red = ${}^{14}C_1$ = 14

Thus, required probability = $\frac{14}{19}$

22. (a) : Let the first chosen square is white. In this case total number of squares that don't belong to the row or column associated with the selected square is 49. Out of these 25 are white and 24 are black.

Thus, corresponding probability $=\frac{1}{2}\cdot\frac{25}{49}$

Similarly, if the first square is black, then the corresponding probability $=\frac{1}{2}\cdot\frac{25}{49}$

Thus, required probability $=\frac{1}{2}\left(\frac{25}{49}+\frac{25}{49}\right)=\frac{25}{49}$

23. (a) : Following equally likely outcomes may occur when one of the dice show four {(4, 1), (1, 4), (4, 2), (2, 4), (4, 3), (3, 4), (4, 4), (4, 5), (5, 4), (4, 6), (6, 4)} Out of these eleven outcomes exactly 2 outcomes favor the cause of second dice showing a six.

Thus, required probability $=\frac{2}{11}$

24. (b) : In the case end digit of the product should be 5. Now, if end digit of each number is either 1, 3, 5, 7 or 9 then product of these numbers will end with either 1, 3, 5, 7 or 9.
On the other hand, end digit of the product will be 1, 3, 7, or 9, if end digit of each integer is either 1, 3, 7 or 9. Thus, total ways in which the product ends with 5 is $5^4 - 4^4$, also total ways of selecting the end digit of the number are 10.

Thus, required probability $=\frac{369}{10^4}$

25. (c) : $P(\bar{A}/\bar{B})=\frac{P(\bar{A}\cap\bar{B})}{P(\bar{B})}=\frac{1-P(A\cup B)}{P(\bar{B})}$

26. (a) : Person will laugh if the selected number is (43, 34, 62, 26).
Thus, probability that person will not laugh on seeing the number $=\frac{86}{90}=\frac{43}{45}$

Thus, required probability $=1-\left(\frac{43}{45}\right)^3$

27. (d) : Total number of ways in which $n_1 + n_2 = 100$ is equal to 99.
Now, $n_1 \cdot n_2 > 1600 \Rightarrow n_1(100 - n_1) > 1600$
$\Rightarrow\ n_1^2 - 100n_1 + 1600 < 0 \Rightarrow (n_1 - 80)(n_1 - 20) < 0$
$\Rightarrow\ 20 < n_1 < 80 \Rightarrow 21 \le n_1 \le 79$
Thus, number of favourable ways = 79 – 21 + 1 = 59

Hence, required probability $=\frac{59}{99}$

28. (b) : Roots will be non real, if $b^2 < 4ac$.
Now, if 'b' is one then $4ac$ is definitely greater than one (that is 'a' and 'c' can be chosen in 4 ways).
If $b = 2$ and $b^2 < 4ac$, then 'a' and 'c' both at the same times should not be equal to one.
That is for $b = 2$, we must have
$a = 1, c = 2$ or $a = 2, c = 1$ or $a = 2, c = 2$
(*i.e.*, 'a' and 'c' can be chosen in three ways)

Thus, required probability = $\frac{7}{8}$.

29. (c) : There are three mutually exclusive cases :
(i) 2 are from A and another is either from B or C.

Corresponding probability $=\frac{{}^2C_2\cdot({}^3C_1+{}^4C_1)}{{}^9C_3}=\frac{7}{84}$

(ii) 2 are from B and another is either from A or C.

Corresponding probability $=\frac{{}^3C_2\cdot({}^2C_1+{}^4C_1)}{{}^9C_3}=\frac{18}{84}$

(iii) 2 are from C and another is either from A or B.
Corresponding probability

$=\frac{{}^4C_2.({}^2C_1+{}^3C_1)}{{}^9C_3}=\frac{30}{84}$

Thus, required probability $=\frac{7+18+30}{84}=\frac{55}{84}$

30. (d) : Total number of ways of dialing the last three digits such that exactly 2 of them are same

$= {}^{10}C_2\cdot 2\cdot\frac{3!}{2!}=270$

Thus, required probability $=\frac{1}{270}$

31. (c) : Let 'A' has 'r' elements, then 'B' have all the remaining $(n - r)$ elements and none of the elements that are already present in A.

Thus, total number of favourable ways $=\sum_{r=0}^{n}{}^nC_r=2^n$

Hence, required probability $=\frac{2^n}{4^n}=\frac{1}{2^n}$

MPP-9 CLASS XII ANSWER KEY

1.	(b)	2.	(a)	3.	(a)	4.	(c)	5.	(c)
6.	(c)	7.	(c)	8.	(a,c)	9.	(a,c)	10.	(a,d)
11.	(b,d)	12.	(a, b)	13.	(a,c,d)	14.	(d)	15.	(b)
16.	(a)	17.	(7)	18.	(3)	19.	(5)	20.	(1)

32. (b) : Total number of functions = 5^3

Now let us count the favourable number of functions.

Let '1' is assigned to r, where $r = 1, 2, 3, 4, 5$.

Now '2' can be assigned to $r, r+1, \ldots\ldots 5$. Let '2' is assigned to j, where $j = r, r+1, \ldots. 5$.

Finally, '3' can be assigned in $(6-j)$ ways.

Thus, total number of favourable ways

$$=\sum_{r=1}^{5}\left(\sum_{j=r}^{5}(6-j)\right)=\sum_{r=1}^{5}\frac{(6-r)(7-r)}{2}$$

$$=\frac{1}{2}\sum_{r=1}^{5}(42-13r+r^2)=35$$

Thus, required probability $=\frac{35}{5^3}=\frac{7}{25}$

33. (b) : Total outcomes = 2^n

Total number of favourable outcomes

$= {}^nC_0 + {}^nC_2 + {}^nC_4 + \ldots\ldots {}^nC_{n/2} = 2^{n-1}$

Thus, required probability $=\frac{2^{n-1}}{2^n}=\frac{1}{2}$

34. (c) : $P(n_1 \le r / n_2 \le r)=\frac{P(n_1 \le r \cap n_2 \le r)}{P(n_2 \le r)}$

$$=\frac{{}^rC_2 / {}^nC_2}{r/n}=\frac{r-1}{n-1}$$

35. (a) : Probability of winning the prize in single game

$$=\frac{25}{25^2}=\frac{1}{25}$$

In this case first 4 games, must result in exactly two prizes and 5^{th} game must result in prize. Thus, required probability

$$={}^4C_2\left(\frac{1}{25}\right)^2\cdot\left(\frac{24}{25}\right)^2\cdot\frac{1}{25}=\frac{6\cdot(24)^2}{(25)^5}$$

36. (c) : E_1 : There is a gust of wind.

E_2 : There is no gust of wind.

A : Archer misses the target.

$P(A) = P(E_1) \,.\, P(A/E_1) + P(E_2) \,.\, P(A/E_2)$

$$=\frac{3}{10}\cdot\frac{3}{5}+\frac{7}{10}\cdot\frac{3}{10}=\frac{39}{100}$$

Now, required probability $=P(E_2 / A)=\frac{P(E_2 \cap A)}{P(A)}$

$$=\frac{P(E_2)\cdot P(A/E_2)}{P(A)}=\frac{\frac{7}{10}\cdot\frac{3}{10}}{\frac{39}{100}}=\frac{7}{13}$$

37. (c) : Let A's throw is r, ($r = 1, 2, 3, \ldots\ldots, 6$), then B's outcome can be ($r, r+1, \ldots\ldots, 6$)

Thus, required probability $=\sum_{r=1}^{6}\frac{7-r}{36}=\sum_{r=1}^{6}\frac{r}{36}=\frac{7}{12}$

38. (a) : $P(E_1)=\frac{3}{6}=\frac{1}{2}$; $P(E_2)=\frac{3}{6}\cdot\frac{3}{6}=\frac{1}{4}$;

$$P(E_3)=\frac{6}{6^3}=\frac{1}{36}$$

Now, $P(E_1 \cap E_2)$ = Probability of getting even number on each throw $=\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{8}$ $= P(E_1) \cdot P(E_2)$

$P(E_1 \cap E_3)$ = Probability of getting even number on each throw $=\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{8} \ne P(E_1)\cdot P(E_3)$

$P(E_2 \cap E_3)$ = Probability of getting even number on each throw $=\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{8}\ne P(E_2)\cdot P(E_3)$

39. (d) : $P(A \cup B) \ge$ maximum $\{P(A),\ P(B)\}$ $=\frac{2}{3}$

$P(A \cap B') \le$ minimum $\{P(A), P(B')\}$ $=\frac{1}{3}$

$P(A \cap B) \ge 0$

and $P(A \cap B) \le$ minimum $\{P(A), P(B)\}$ $=\frac{1}{2}$

Thus, all the statements are correct.

40. (c) : Let the selected numbers be n_1, n_2 and n_3.

We must have $2n_2 = n_1 + n_3$.

Thus, $n_1 + n_3$ must be even. That means n_1 and n_3 both must have same nature (either odd or even).

Thus, required probability $=\frac{2\cdot{}^{12}C_2}{{}^{24}C_3}=\frac{3}{46}$

CLASS XII | Series 9

ACE YOUR WAY CBSE

Linear Programming and Probability

Important Formulae

LINEAR PROGRAMMING

TERMINOLOGIES RELATED TO L.P.P.

Term	*Definition*
Constraints	*The restrictions or linear inequalities or equations in the variables of an L.P.P., which describe the conditions under which the optimisation is to be accomplished.*
Non-negativity Constraints	*The assumption that negative values of variables are not possible in the solution. They are described as $x \geq 0, y \geq 0$*
Objective Function	*The linear function which is to be maximised or minimised under given constraints.*
Feasible Region and Feasible Solution	*The common region determined by all the constraints including non-negative constraints of a L.P.P. is called the feasible region or solution region. Feasible region is always a convex set. Feasible region may be bounded or unbounded.*
Infeasible Region and Infeasible Solution	*The region other than the feasible region is called infeasible region of the L.P.P. and the points which come under infeasible region are called infeasible solution.*

FORMULATION OF A L.P.P.

- *The three steps in the mathematical formulation of an L.P.P. are as follows:*
  - *(i) Identify the objective function as a linear combination of variables (x and y) and construct all constraints i.e.linear equations and inequations involving these variables. Thus, an L.P.P. can be stated mathematically as*
    Maximise (or minimise) $z = ax + by$
    Subject to the constraints:
    $a_ix + b_iy \leq$ (or $\geq$ or $=$ or $>$ or $<$) c_i, where $i = 1$ n, $x \geq 0$, $y \geq 0$ (non-negative constraints).
  - *(ii) Find the solutions (feasible region) of these equations and inequations by some mathematical method.*
  - *(iii) Find an optimal solution i.e. select particular values of the variables x and y that give the desired value (maximum/minimum) of the objective function.*

PROBABILITY

CONDITIONAL PROBABILITY

- *Let A and B be two events associated with a random experiment. Then, the probability of occurrence of A under the condition that B has already occurred and $P(B) \neq 0$, is called the conditional probability.*
  i.e., $P(A/B) = \dfrac{P(A \cap B)}{P(B)}$

- *If A and B are two events associated with a random experiment, then*
  $P(A \cap B) = P(A) \cdot P(B/A)$, if $P(A) \neq 0$
  $P(A \cap B) = P(B) \cdot P(A/B)$, if $P(B) \neq 0$
- *Events are said to be independent, if the occurrence or non-occurrence of one does not affect the probability of the occurrence of the other.*
  i.e., $P(A/B) = P(A)$, $P(B) \neq 0$; $P(B/A) = P(B)$, $P(A) \neq 0$
- *Two events A and B are independent events associated with a random experiment, if*
  $P(A \cap B) = P(A) \cdot P(B)$
- *If* $A_1, A_2, A_3, \ldots A_n$, are n independent events, then $P(A_1 \cup A_2 \ldots\ldots \cup A_n) = 1 - P(\bar{A}_1) \cdot P(\bar{A}_2) \ldots\ldots P(\bar{A}_n)$
- *If A is any event which occurs with* E_1 *or* E_2 *....or* E_n*, then* $P(A) = P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2) + \ldots + P(E_n)\,P(A/E_n)$
- *Baye's theorem : If A is any event which occurs with* E_1 *or* E_2 *.... or* E_n*, then*
  $$P(E_i/A) = \frac{P(E_i) \cdot P(A/E_i)}{\sum_{i=1}^{n} P(E_i) \cdot P(A/E_i)},\ i = 1, 2, \ldots, n$$
- *A real valued function X, defined on a sample space S, of a random experiment i.e.,* $X : S \to R$ *is called a random variable.*
- *Binomial Distribution : The probability of 'r' success in n trials is* ${}^nC_r \cdot p^r \cdot q^{n-r}$*, where p is the probability of success and 'q' is the probability of failure in a single trial.*
- *Mean of Binomial Distribution = np*
  Variance = npq, standard deviation= $\sqrt{npq}$
- *Let X be a random variable whose possible values* $x_1, x_2, x_3, \ldots, x_n$ *occur with probabilities* $p_1, p_2, p_3, \ldots, p_n$ *respectively. Then we define expectation of X, denoted by E(X) and is given by* $E(X) = \sum_{i=1}^{n} x_i p_i$.
- $Var(X) = E(X^2) - [E(X)]^2$

WORK IT OUT

VERY SHORT ANSWER TYPE

1. If a die is tossed, then the sample space $S = \{1, 2, 3, 4, 5, 6\}$. Let $A = \{1, 3, 5\}$, $B = \{2, 4, 6\}$ and $C = \{2, 3, 5\}$. Find : (i) $P(A/C)$ (ii) $P(C/B)$.

2. Find the maximum value of $z = 4x + 3y$ subject to $x + y \leq 4$, $x, y \geq 0$.

3. If $P(A) = 0.4$, $P(B) = 0.8$, $P(B/A) = 0.6$. Find $P(A/B)$ and $P(A \cup B)$.

4. If A and B are mutually exclusive events, find $P(A/B)$.

5. X is taking up subjects-Mathematics, Physics and Chemistry in the examination. His probabilities of getting grade A in these subjects are 0.2, 0.3 and 0.5 respectively. Find the probability that he gets grade A in all subjects.

SHORT ANSWER TYPE

6. Consider $z(x, y) = px + qy$ subject to $2x + y \leq 10$, $x + 3y \leq 15$, $x, y \geq 0$. If z is maximum at both the points (3, 4) and (0, 5), then find q in terms of p.

7. A bag contains 3 red and 7 black balls. Two balls are selected at random one-by-one without replacement. If the second selected ball happens to be red, what is the probability that the first selected ball is also red?

8. If A and B are independent events, prove that $P(A \cup B) \cdot P(A' \cap B') \leq P(C)$, where C is an event defined that exactly one of A and B occurs.

9. Find the minimum value of $z = 2x + y$ subject to : $x + 2y \geq 3$, $2x + y \geq 6$, $x, y \geq 0$.

10. The random variable X can take only the values 0, 1, 2. Given that $P(X = 0) = P(X = 1) = p$ and $E(X^2) = E(X)$, find the value of p.

LONG ANSWER TYPE-I

11. Find the maximum value of $z = x + y$ subject to $x + y \leq 3$, $-2x + y \leq 1$, $x \leq 2$, $x, y \geq 0$.

12. The ratio of the number of boys to the number of girls in a class is 1 : 2. It is known that the probabilities of a girl and a boy getting a first division are 0.25 and 0.28 respectively. Find the probability that student chosen at random will get first division.

13. A brick manufacturer has two depots A and B, with stocks of 30000 and 20000 bricks respectively. He receives orders from three builders P, Q and R for 15000, 20000 and 15000 bricks respectively. The cost (in ₹) of transporting 1000 bricks to the builders from the depots are as given in the table :

From \ To	Transportation cost per 1000 bricks (in ₹)		
	P	**Q**	**R**
A	40	20	20
B	20	60	40

The manufacturer wishes to fulfil the order so that transportation cost is minimum. Formulate the LPP.

14. There are 4 white and 3 black balls in a box. In another box there are 3 white and 4 black balls. An unbiased dice is rolled. If it shows a number less than or equal to 3, then a ball is drawn from the first box but if it shows a number more than 3 then a ball is drawn from the second box. If the ball drawn is black, then find the probability that the ball was drawn from the first box.

15. A bag X contains 3 white and 2 black balls; another bag Y contains 2 white and 4 black balls. A bag and a ball out of it is picked at random. What is the probability that the ball is white?

LONG ANSWER TYPE-II

16. Kellogg is a new cereal formed of a mixture of bran and rice that contains at least 88 grams of protein and at least 36 milligrams of iron. Knowing that bran contains 80 grams of proteins and 40 milligrams of iron per kilogram, and that rice contains 100 grams of protein and 30 milligrams of iron per kilogram, find the minimum cost of producing this new cereal if bran costs ₹ 5 per kilogram and rice costs ₹ 4 per kilogram.

17. The sum of mean and variance of a binomial distribution is 15 and the sum of their squares is 117. Determine the distribution.

18. A can hit a target 4 times in 5 shots, B 3 times in 4 shots, and C 2 times in 3 shots. Calculate the probability that

(i) B, C may hit and A may not.

(ii) Any two of A, B and C will hit the target

(iii) none of them will hit the target

19. A manufacturer makes two products, A and B. Product A sells at ₹ 200 per unit and takes 30 minutes to make. Product B sells at ₹ 300 per unit and takes 1 hour to make. There is a permanent order of 14 units of product A and 16 units of product B. A working week consists of 40 hours of production and the weekly turnover must not be less than ₹ 10000. If the profit on each of product A is ₹ 20 and on product B is ₹ 30, then how many product A and B respectively should be produced so that the profit is maximum? Also, find the maximum profit.

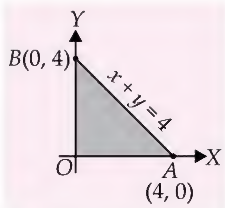
20. Sixteen players $P_1, P_2, ..., P_{16}$ play in a tournament. They are divided into eight pairs at random. From each pair a winner is decided on the basis of a game played between the two players of the pairs. Assuming that all the players are of equal strength, find the probability that exactly one of the two players P_1 and P_2 is among the eight winners.

SOLUTIONS

1. (i) $P(A/C) = \dfrac{P(A \cap C)}{P(C)} = \dfrac{P\{3, 5\}}{1/2} = \dfrac{2}{6} \cdot \dfrac{2}{1} = \dfrac{2}{3}$

(ii) $P(C/B) = \dfrac{P(C \cap B)}{P(B)} = \dfrac{P\{2\}}{1/2} = \dfrac{1}{6} \cdot \dfrac{2}{1} = \dfrac{1}{3}$

2. The corners of the feasible region are $O(0, 0)$, $A(4, 0)$, $B(0, 4)$.



$\because \; z = 4x + 3y$

$\Rightarrow \; z(O) = 0, z(A) = 16$

$z(B) = 12$

$\therefore$ Maximum value is $z = 16$ at A.

3. $P(A \cap B) = P(A) \cdot P(B/A) = (0.4)(0.6) = 0.24$

Now, $P(A / B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{0.24}{0.8} = 0.3$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$
$= 0.4 + 0.8 - 0.24 = 0.96$

4. Since A and B are mutually exclusive events,

$\therefore \; A \cap B = \phi$.

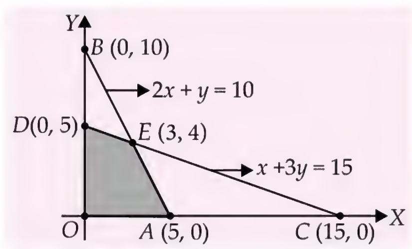
Hence, $P(A / B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{P(\phi)}{P(B)} = 0$

5. Let M, P' and C denote the events that X gets grade A in Mathematics, Physics and Chemistry respectively.

Given, $P(M) = 0.2$, $P(P') = 0.3$, $P(C) = 0.5$

$\therefore \; P(\text{grade } A \text{ in all subjects}) = P(M \cap P' \cap C)$
$= P(M) \cdot P(P') \cdot P(C) = 0.2 \times 0.3 \times 0.5 = 0.03$

6. The feasible region is shown in the graph.



Since $z = px + qy$

$\therefore\ \ z(O) = 0, z(A) = 5p, z(E) = 3p + 4q, z(D) = 5q$

Also, $z(E) = z(D) \Rightarrow 3p + 4q = 5q \quad \therefore \quad q = 3p.$

7. Let A be the event of drawing a red ball in first draw and B be the event of drawing a red ball in second draw.

$\therefore\ \ P(A) = \dfrac{{}^3C_1}{{}^{10}C_1} = \dfrac{3}{10}$

Now, $P(B/A)$ = Probability of drawing a red ball in the second draw, when a red ball already has been drawn in the first draw $= \dfrac{{}^2C_1}{{}^9C_1} = \dfrac{2}{9}$

$\therefore$ Required probability $= P(A \cap B) = P(A) \cdot P(B/A)$

$= \dfrac{3}{10} \times \dfrac{2}{9} = \dfrac{1}{15}$

8. L.H.S. $= P(A \cup B) \cdot P(A' \cap B')$

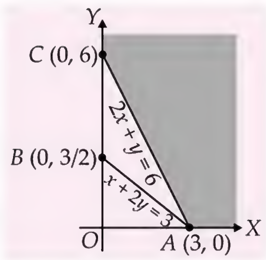
$= P(A \cup B) \cdot P(A') \cdot P(B') \le (P(A) + P(B))\, P(A') \cdot P(B')$

$= P(A) \cdot P(A') \cdot P(B') + P(B) \cdot P(A') \cdot P(B')$

$\le P(A)P(B') + P(B)\, P(A')$ [since $P(A') \le 1, P(B') \le 1$]

$= P(A \cap B') + P(A' \cap B) = P(C) =$ R.H.S.

9. The feasible region is unbounded.



$\therefore\ \ z(A) = 2 \times 3 = 6$

$z(C) = 6$

$\therefore$ Minimum value of z is 6 on the line $2x + y = 6$.

Maximum value of z does not exist.

10. Clearly, $P(X = 0) + P(X = 1) + P(X = 2) = 1$

$\Rightarrow p + p + P(X = 2) = 1 \quad \Rightarrow \quad P(X = 2) = 1 - 2p.$

So, the probability distribution of X is as given below :

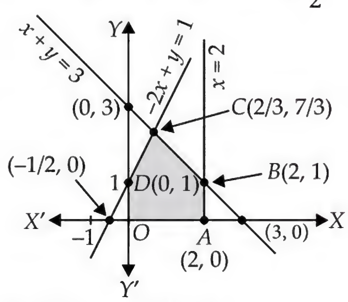
x_i	0	1	2
p_i	p	p	$1 - 2p$

$\therefore\ \ E(X) = (0 \times p) + (1 \times p) + 2\,(1 - 2p) = 2 - 3p$

and $E(X^2) = 0^2 \times p + 1^2 \times p + 2^2(1 - 2p) = 4 - 7p$

Now, $E(X^2) = E(X) \Rightarrow 4 - 7p = 2 - 3p \Rightarrow p = \dfrac{1}{2}.$

11. The corners points of feasible region are $O(0, 0)$, $A(2, 0)$, $B(2, 1)$, $C\left(\dfrac{2}{3}, \dfrac{7}{3}\right)$, $D(0, 1)$.



Now, $z = x + y$

$\Rightarrow\ z(A) = 2,$

$z(B) = 3, z(C) = 3, z(D) = 1, z(O) = 0$

$\therefore$ Maximum $z = 3$ along line BC.

12. Let E_1 = event that a student selected at random is a girl and E_2 = event that a student selected at random is a boy.

E = event that a student selected at random will get first division

Given, Number of boys : Number of girls = 1 : 2

$\therefore\ \ P(E_1) = \dfrac{2}{3}$ and $P(E_2) = \dfrac{1}{3}$

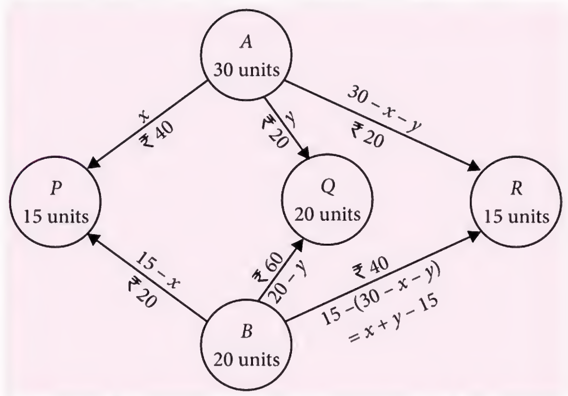
By rule of total probability

$P(E) = P(E_1) \cdot P(E/E_1) + P(E_2) \cdot P(E/E_2)$

$= \dfrac{2}{3}(0.25) + \dfrac{1}{3}(0.28) = \dfrac{0.78}{3} = 0.26$

13. The given information can be expressed as given in the diagram.

We assume that 1 unit = 1000 bricks



Suppose that depot A supplies x units to P and y units to Q, so that depot A supplies $(30 - x - y)$ units of bricks to builder R.

Now as P requires a total of 15000 bricks, it requires $(15 - x)$ units from depot B.

Similarly Q requires $(20 - y)$ units from B and R requires $15 - (30 - x - y) = x + y - 15$ units from B.

Total transportation cost,

$Z = 40x + 20y + 20(30 - x - y) + 20(15 - x)$
$\qquad + 60(20 - y) + 40(x + y - 15)$

$= 40x - 20y + 1500$

Obviously the constraints are that all quantities of bricks supplied from A and B to P, Q, R are non-negative.

$\therefore\ \ x \ge 0, y \ge 0, 30 - x - y \ge 0, 15 - x \ge 0, 20 - y \ge 0,$
$x + y - 15 \ge 0$

Since, 1500 is a constant, hence instead of minimizing $Z = 40x - 20y + 1500$, we minimize $Z = 40x - 20y$.

Hence, mathematical formulation of the given LPP is

Minimize $Z = 40x - 20y$,

subject to the constraints : $x + y \ge 15$, $x + y \le 30$, $x \le 15$, $y \le 20$, $x \ge 0$, $y \ge 0$

14. Let E_1 = Event of drawing a ball from the first box.
E_2 = Event of drawing a ball from the second box.
B = Event of drawing a black ball.

$\therefore \quad P(E_1) = \frac{3}{6} = \frac{1}{2}$ and $P(E_2) = \frac{3}{6} = \frac{1}{2}$

Clearly, E_1 and E_2 are mutually exclusive and exhaustive.

Now, $P\left(\frac{B}{E_1}\right) = \frac{3}{7}$ and $P\left(\frac{B}{E_2}\right) = \frac{4}{7}$.

By Bayes' theorem, required probability

$$= P\left(\frac{E_1}{B}\right) = \frac{P(E_1)\cdot P\left(\frac{B}{E_1}\right)}{P(E_1)\cdot P\left(\frac{B}{E_1}\right) + P(E_2)\cdot P\left(\frac{B}{E_2}\right)}$$

$$= \frac{\frac{1}{2}\cdot\frac{3}{7}}{\frac{1}{2}\cdot\frac{3}{7} + \frac{1}{2}\cdot\frac{4}{7}} = \frac{3}{7}$$

15. Let E_1 = the event of selecting bag X
E_2 = the event of selecting bag Y
E = the event of drawing a white ball

$\therefore \quad P(E_1) = \frac{1}{2},\ P(E_2) = \frac{1}{2}$

Let $A = E_1 \cap E$ and $B = E_2 \cap E$

$\therefore$ Required probability, $P(E) = P(A \cup B) = P(A) + P(B)$

[Since events A and B are mutually exclusive]

$= P(E_1 \cap E) + P(E_2 \cap E)$

$= P(E_1) \cdot P(E/E_1) + P(E_2) \cdot P(E/E_2)$

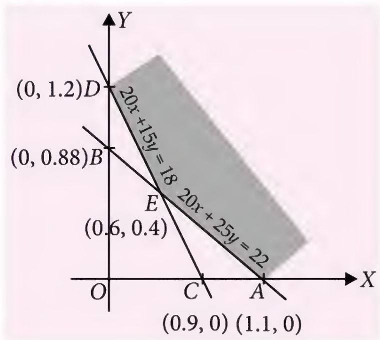
$$= \frac{1}{2}\cdot\frac{3}{5} + \frac{1}{2}\cdot\frac{2}{6} = \frac{7}{15}$$

16. Let the cereal contain x kg of bran and y kg of rice.
Let z be the cost of the new cereal.
According to question,

$$x \times \frac{80}{1000} + y \times \frac{100}{1000} \ge \frac{88}{1000} \text{ or } 20x + 25y \ge 22$$

$$x \times \frac{40}{1000000} + y \times \frac{30}{1000000} \ge \frac{36}{1000000} \quad \text{or } 20x + 15y \ge 18$$

$x \ge 0,\ y \ge 0$



Mathematical formulation of the LPP is :
Minimize $z = 5x + 4y$
subject to constraints :

$$20x + 25y \ge 22,\ 20x + 15y \ge 18,\ x \ge 0,\ y \ge 0$$

The corner points of unbounded feasible region as $A(1.1, 0)$, $E(0.6, 0.4)$ and $D(0, 1.2)$

At $A(1.1, 0)$: $z = 5 \times 1.1 + 4 \times 0 = 5.5$
At $E(0.6, 0.4)$: $z = 5 \times 0.6 + 4 \times 0.4 = 4.6$
At $D(0, 1.2)$: $z = 5 \times 0 + 4 \times 1.2 = 4.8$

$\therefore$ The minimum cost of producing this cereal is ₹ 4.6

17. Let n and p be the parameters of the distribution.

$\therefore$ Mean = np and Variance = npq.

Given, $np + npq = 15$ and $n^2p^2 + n^2p^2q^2 = 117$

$\Rightarrow np(1 + q) = 15$ and $n^2p^2(1 + q^2) = 117$

$\Rightarrow n^2p^2(1 + q)^2 = 225$ and $n^2p^2(1 + q^2) = 117$

$$\Rightarrow \frac{n^2p^2(1+q)^2}{n^2p^2(1+q^2)} = \frac{225}{117} \quad \Rightarrow \frac{(1+q)^2}{(1+q^2)} = \frac{225}{117}$$

$$\Rightarrow \frac{1+q^2+2q}{1+q^2} = \frac{225}{117} \quad \Rightarrow 1 + \frac{2q}{1+q^2} = \frac{225}{117}$$

$$\Rightarrow \frac{1+q^2}{2q} = \frac{13}{12} \quad \Rightarrow \frac{1+q^2+2q}{1+q^2-2q} = \frac{13+12}{13-12}$$

$$\Rightarrow \left(\frac{1+q}{1-q}\right)^2 = 25 \quad \Rightarrow \frac{1+q}{1-q} = 5 \quad \Rightarrow 6q = 4$$

$$\Rightarrow q = \frac{2}{3} \quad \therefore \quad p = 1 - q = 1 - \frac{2}{3} = \frac{1}{3}$$

Putting $p = \frac{1}{3},\ q = \frac{2}{3}$ in $np + npq = 15$, we get

$$\frac{n}{3} + \frac{2n}{9} = 15 \quad \Rightarrow \frac{5n}{9} = 15 \quad \Rightarrow n = 27$$

Thus, $n = 27$, $p = \frac{1}{3}$ and $q = \frac{2}{3}$

Hence, the distribution is given by

$$P(X = r) = {}^{27}C_r \left(\frac{1}{3}\right)^r \left(\frac{2}{3}\right)^{27-r},\ r = 0, 1, 2, ..., 27.$$

18. Consider the following events :
E = A hits the target, F = B hits the target and G = C hits the target

We have, $P(E) = \frac{4}{5}$, $P(F) = \frac{3}{4}$ and $P(G) = \frac{2}{3}$

(i) $P(B, C$ may hit and A may not$) = P(\bar{E} \cap F \cap G)$
$= P(\bar{E})\ P(F)\ P(G)$

[$\because$ E, F, G are independent events]

$$= \left(1 - \frac{4}{5}\right) \times \frac{3}{4} \times \frac{2}{3} = \frac{1}{10}$$

(ii) P (Any two of A, B and C will hit the target)

$= P(E \cap F \cap \overline{G}) \cup (\overline{E} \cap F \cap G) \cup (E \cap \overline{F} \cap G))$

$= P(E \cap F \cap \overline{G}) + P(\overline{E} \cap F \cap G) + P(E \cap \overline{F} \cap G))$

$= P(E)\, P(F)\, P(\overline{G}) + P(\overline{E})\, P(F)\, P(G) + P(E)\, P(\overline{F})\, P(G)$

$$= \left(\frac{4}{5}\times\frac{3}{4}\times\frac{1}{3}\right)+\left(\frac{1}{5}\times\frac{3}{4}\times\frac{2}{3}\right)+\left(\frac{4}{5}\times\frac{1}{4}\times\frac{2}{3}\right)=\frac{13}{30}$$

(iii) P(None of A, B and C will hit the target)

$$= P(\overline{E} \cap \overline{F} \cap \overline{G}) = \frac{1}{5}\times\frac{1}{4}\times\frac{1}{3}=\frac{1}{60}$$

19. Let x be the number of units of the product A and y be the number of units of the product B produced. Let z be the total profit.

$\therefore\ z = 20x + 30y$

x and y must satisfy the following conditions

$200x + 300y \geq 10000$ or $2x + 3y \geq 100$

$\frac{1}{2}x + 1\cdot y \leq 40$ or $x + 2y \leq 80$

$x \geq 14,\ y \geq 16,\ x \geq 0, y \geq 0$

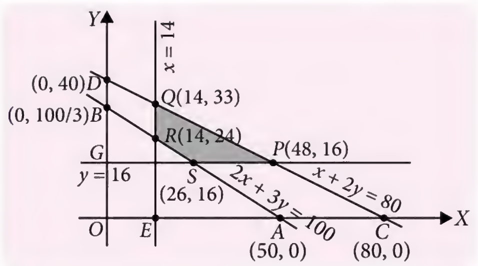
The mathematical formulation of the LPP will be

Maximize $z = 20x + 30y$

subject to constraints :

$2x + 3y \geq 100, x + 2y \leq 80, x \geq 14, y \geq 16, x \geq 0, y \geq 0$

On plotting, we get $S(26, 16)$, $P(48, 16)$, $Q(14, 33)$ and $R(14, 24)$ as corners of feasible region.



$\because$ Maximize $z = 20x + 30y$

$\therefore$ At $S(26, 16)$, $z = 1000$

At $P(48, 16)$, $z = 1440$

At $Q(14, 33)$, $z = 1270$

At $R(14, 24)$, $z = 1000$

Hence, the maximum profit is ₹1440 and it is attained when 48 units of product A and 16 units of product B are produced.

20. Let E_1 and E_2 denote the event that P_1 and P_2 are paired and not paired together and let A denote the event that one of two players P_1 and P_2 is among the winners.

Since P_1 can be paired with any of the remaining 15 players.

We have, $P(E_1) = \frac{1}{15}$

and $P(E_2) = 1 - P(E_1) = 1 - \frac{1}{15} = \frac{14}{15}$.

In case E_1 occurs, it is certain that one of P_1 and P_2 will be among the winners. Let C and D be the events that P_1 and P_2 wins respectively. In case E_2 occurs, the probability that exactly one of P_1 and P_2 is among the winners is

$P\{(C \cap \overline{D}) \cup (\overline{C} \cap D)\} = P(C)\, P(\overline{D}) + P(\overline{C})\, P(D)$

$$= \left(\frac{1}{2}\right)\left(1-\frac{1}{2}\right)+\left(1-\frac{1}{2}\right)\left(\frac{1}{2}\right)=\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$$

i.e., $P(A/E_1) = 1$ and $P(A/E_2) = \frac{1}{2}$

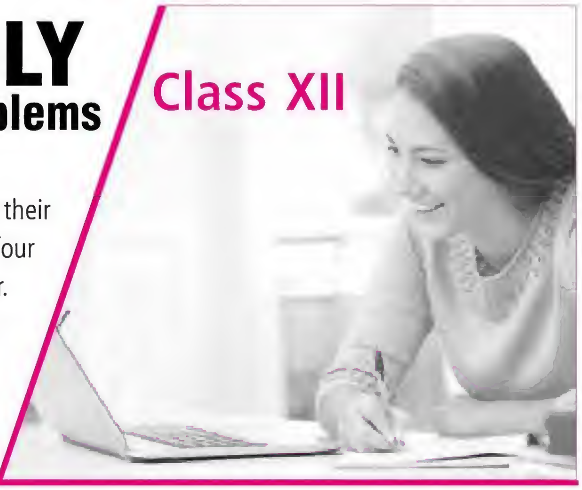
By the total probability rule,

$$P(A) = P(E_1)\cdot P\left(\frac{A}{E_1}\right) + P(E_2)\cdot P\left(\frac{A}{E_2}\right)$$

$$= \frac{1}{15}(1) + \frac{14}{15}\left(\frac{1}{2}\right) = \frac{8}{15}$$

MPP-9 MONTHLY Practice Problems

Class XII



This specially designed column enables students to self analyse their extent of understanding of specified chapters. Give yourself four marks for correct answer and deduct one mark for wrong answer. Self check table given at the end will help you to check your readiness.

Vector Algebra and Three Dimensional Geometry

Total Marks : 80 **Time Taken : 60 Min.**

Only One Option Correct Type

1. A unit tangent vector at $t = 2$ on the curve $x = t^2 + 2, y = 4t - 5, z = 2t^2 - 6t$ is
   (a) $\frac{1}{\sqrt{3}}(\hat{i} + \hat{j} + \hat{k})$ (b) $\frac{1}{3}(2\hat{i} + 2\hat{j} + \hat{k})$
   (c) $\frac{1}{\sqrt{6}}(2\hat{i} + \hat{j} + \hat{k})$ (d) none of these

2. The projection of a line segment on the coordinates axes are 2, 3, 6. Then, the length of the line segment is
   (a) 7 (b) 5 (c) 1 (d) 11

3. The angle between a line with direction ratios proportional to 2, 2, 1 and a line joining (3, 1, 4) to (7, 2, 12), is
   (a) $\cos^{-1}\left(\frac{2}{3}\right)$ (b) $\cos^{-1}\left(-\frac{2}{3}\right)$
   (c) $\tan^{-1}\left(\frac{2}{3}\right)$ (d) none of these

4. Let $\vec{a}, \vec{b}, \vec{c}$ be three non-coplanar vectors and $\vec{d}$ be a non-zero vector which is perpendicular to $\vec{a} + \vec{b} + \vec{c}$. If $\vec{d} = x\vec{b} \times \vec{c} + y\vec{c} \times \vec{a} + z\vec{a} \times \vec{b}$, then
   (a) $xy + yz + zx = 0$ (b) $x = y = z$
   (c) $x^3 + y^3 + z^3 = 3xyz$ (d) $x + y + z = 1$

5. The line through the points $(5, 1, a)$ and $(3, b, 1)$ crosses yz plane at the point $\left(0, \frac{17}{2}, \frac{-13}{2}\right)$. Then
   (a) $a = 2, b = 8$ (b) $a = 4, b = 6$
   (c) $a = 6, b = 4$ (d) $a = 8, b = 2$

6. If $(\vec{a}\times\vec{b})^2 + (\vec{a}\cdot\vec{b})^2 = 144$ and $|\vec{a}| = 4$, then $|\vec{b}|$ is equal to
   (a) 16 (b) 8 (c) 3 (d) 12

One or More Than One Option(s) Correct Type

7. Let $A_1, A_2, ..., A_n$ $(n > 2)$ be the vertices of a regular polygon of n sides with its centre at the origin. Let $\vec{a}_k$ be the position vector of the point A_k, $k = 1, 2, ..., n$. If $\left|\sum_{k=1}^{n-1}(\vec{a}_k \times \vec{a}_{k+1})\right| = \left|\sum_{k=1}^{n-1}(\vec{a}_k \cdot \vec{a}_{k+1})\right|$, then the minimum value of n is
   (a) 1 (b) 2 (c) 8 (d) 9

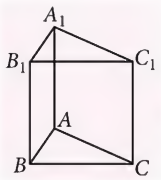
8. Let P_1 denote the equation of the plane to which the vector $(\hat{j} + \hat{i})$ is normal and which contains the line L whose equation is $\vec{r} = \hat{i} + \hat{j} + \hat{k} + \lambda(\hat{i} - \hat{j} - \hat{k})$. Let P_2 denote the equation of the plane containing the line L and a point with position vectors $\hat{j}$. Which of the following holds goods?
   (a) Equation of P_1 is $x + y = 2$
   (b) Equation of P_2 is $\vec{r}\cdot\left(\hat{i} - 2\hat{j} + \hat{k}\right) = 2$.
   (c) The acute angle between P_1 and P_2 is $\cot^{-1}\left(\sqrt{3}\right)$.
   (d) The angle between the plane P_2 and the line L is $\tan^{-1}\sqrt{3}$.

9. If $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = \hat{i} - \hat{j}$, then the vectors $(\vec{a}\cdot\hat{i})\hat{i} + (\vec{a}\cdot\hat{j})\hat{j} + (\vec{a}\cdot\hat{k})\hat{k}, (\vec{b}\cdot\hat{i})\hat{i} + (\vec{b}\cdot\hat{j})\hat{j} + (\vec{b}\cdot\hat{k})\hat{k}$, and $\hat{i} + \hat{j} - 2\hat{k}$
   (a) are mutually perpendicular
   (b) are coplanar
   (c) form a parallelopiped of volume 6 units
   (d) form a parallelopiped of volume 3 units

10. The projection of the line $\frac{x-1}{2} = \frac{y+1}{-1} = \frac{z-3}{4}$ on the plane $x + 2y + z = 9$ is the line of intersection of which of the following planes?

(a) $x + 2y + z - 9 = 0$ (b) $9x + 2y + 3z - 4 = 0$
(c) $x - 2y + z + 3 = 0$ (d) $9x - 2y - 5z + 4 = 0$

11. The volume of a right triangular prism $ABCA_1B_1C_1$ is equal to 3. Find the co-ordinates of the vertex A_1, if the co-ordinates of the base vertices of the prism are $A(1, 0, 1)$, $B(2, 0, 0)$ and $C(0, 1, 0)$.



(a) $(-2, 0, 2)$ (b) $(0, -2, 0)$
(c) $(0, 2, 0)$ (d) $(2, 2, 2)$

12. The equations of two lines through the origin which intersect the line $\frac{x-3}{2} = \frac{y-3}{1} = \frac{z}{1}$ at an angle 60° is

(a) $\frac{x}{1} = \frac{y}{2} = \frac{z}{-1}$ (b) $\frac{x}{1} = \frac{y}{-1} = \frac{z}{2}$
(c) $\frac{x}{2} = \frac{y}{1} = \frac{z}{-1}$ (d) $\frac{x}{2} = \frac{y}{-1} = \frac{z}{1}$

13. The line of intersection of the planes $\vec{r} \cdot (\hat{i} + 2\hat{j} + 3\hat{k}) = 0$ and $\vec{r} \cdot (3\hat{i} + 2\hat{j} + \hat{k}) = 0$ is

(a) equally inclined to $\hat{i}$ and $\hat{k}$
(b) equally inclined to $\hat{i}$ and $\hat{j}$
(c) inclined at an angle $\cos^{-1}\sqrt{\frac{2}{3}}$ with $\hat{j}$
(d) inclined at an angle $\cos^{-1}\left(\frac{-1}{\sqrt{6}}\right)$ with $\hat{i}$

Comprehension Type

If $\vec{a}, \vec{b}$ and $\vec{c}$ be any three non-coplanar vectors. Then the system $\vec{a}', \vec{b}'$ and $\vec{c}'$ which satisfies $\vec{a} \cdot \vec{a}' = \vec{b} \cdot \vec{b}' = \vec{c} \cdot \vec{c}' = 1$
and $\vec{a} \cdot \vec{b}' = \vec{a} \cdot \vec{c}' = \vec{b} \cdot \vec{a}' = \vec{b} \cdot \vec{c}' = \vec{c} \cdot \vec{a}' = \vec{c} \cdot \vec{b}' = 0$
is called the reciprocal system to the vectors $\vec{a}, \vec{b}$ and $\vec{c}$.

14. The value of $(\vec{a} \times \vec{a}') + (\vec{b} \times \vec{b}') + (\vec{c} \times \vec{c}')$

(a) $= \vec{a} + \vec{b} + \vec{c}$ (b) $= \vec{a}' + \vec{b}' + \vec{c}'$
(c) $\neq 0$ (d) $= 0$

15. The value of $[\vec{a}'\ \vec{b}'\ \vec{c}']^{-1}$

(a) $< [\vec{a}\ \vec{b}\ \vec{c}]$ (b) $= [\vec{a}\ \vec{b}\ \vec{c}]$
(c) $> [\vec{a}\ \vec{b}\ \vec{c}]$ (d) $= 0$

Matrix Match Type

16. Match the following.

	Column I		Column II
P.	The number of independent constants in the equation of a straight line in 3 dimensions is	1.	2
Q.	A plane passes through $(1, -2, 1)$ and is perpendicular to two planes $2x - 2y + z = 0, x - y + 2z = 4$. If the distance of $(1, 2, 2)$ from it, is $k\sqrt{2}$ units. Then k equals	2.	9
R.	If the lines $\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-1}{4}$ and $\frac{x-3}{1} = \frac{y-k}{2} = \frac{z}{1}$ are coplanar, then $2k$ is	3.	4

	P	Q	R
(a)	3	1	2
(b)	3	2	1
(c)	1	2	3
(d)	2	1	3

Integer Answer Type

17. The distance of the point $\hat{i} + 2\hat{j} + 3\hat{k}$ from the plane $\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = 5$ measured parallel to the vectors $2\hat{i} + 3\hat{j} - 6\hat{k}$ must be

18. The distance of the point (3, 0, 5) from the line $x - 2y + 2z - 4 = 0 = x + 3z - 11$ is

19. If the lines $x - y + z = -1 = 3x - y + az + 1$ and $x + 4y - z - 1 = 0 = x + 2y + bz + 1$ are perpendicular, then $3ab + a - b + c = 0$, where c is

20. Let $\vec{p} = 2\hat{i} - \hat{j} + \hat{k}, \vec{q} = \hat{i} + 2\hat{j} - k$ and $\vec{r} = \hat{i} + \hat{j} - 2\hat{k}$. If $\vec{V} = \vec{p} + \lambda\vec{q}$ and projection of $\vec{V}$ on $\vec{r}$ is $\frac{4}{\sqrt{6}}$, then value of λ is equal to

◆◆

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TARGET JEE

Vectors and 3D Geometry

PROBLEMS

Single Correct Answer Type

1. The ratio in which the plane $\vec{r}\cdot(\hat{i}-2\hat{j}+3\hat{k})=17$ divides the line joining the points $-2\hat{i}+4\hat{j}+7\hat{k}$ and $3\hat{i}-5\hat{j}+8\hat{k}$ is
(a) 3 : 5 (b) 1 : 5 (c) 2 : 7 (d) 3 : 10

2. The length of the perpendicular from origin to the plane passing through a point $\vec{c}$ and containing the line $\vec{r}=\vec{b}+\mu\vec{a}$ is
(a) $\dfrac{[\vec{a}\,\vec{b}\,\vec{c}]}{|\vec{b}\times\vec{c}+\vec{c}\times\vec{a}|}$ (b) $\dfrac{[\vec{c}\,\vec{b}\,\vec{a}]}{|\vec{b}\times\vec{a}+\vec{a}\times\vec{c}|}$
(c) $\dfrac{[\vec{a}\,\vec{b}\,\vec{c}]}{|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}|}$ (d) $\dfrac{[\vec{a}\,\vec{b}\,\vec{c}]}{|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}|}$

3. The position vector of foot of the perpendicular of the point $P(1, 2, 3)$ on the line $(6\hat{i}+7\hat{j}+7\hat{k})+\lambda(3\hat{i}+2\hat{j}-2\hat{k})$ is
(a) $3\hat{i}-5\hat{j}+9\hat{k}$ (b) $3\hat{i}+5\hat{j}-9\hat{k}$
(c) $3\hat{i}+5\hat{j}+9\hat{k}$ (d) none of these

4. A parallelopiped is formed by planes drawn parallel to the co-ordinate axes through the points $A = (2, 3, 4)$ and $B = (11, 9, 8)$. Then, the volume of the parallelepiped is equal to (in cubic units)
(a) 108 (b) 36 (c) 72 (d) 216

5. A line passes through $P(2, 3, 4)$ and having direction ratios (4, 5, 6) meet the plane $x + 2y - 3z + 16 = 0$ at Q then length of PQ is equal to
(a) $\sqrt{936}$ units (b) $\sqrt{693}$ units
(c) $\sqrt{963}$ units (d) $\sqrt{369}$ units

6. If the four faces of a tetrahedron are represented by the equations $\vec{r}\cdot(\beta\hat{i}+\gamma\hat{j})=0$, $\vec{r}\cdot(\gamma\hat{j}+\alpha\hat{k})=0$, $\vec{r}\cdot(\alpha\hat{k}+\beta\hat{i})=0$ and $\vec{r}\cdot(\beta\hat{i}+\gamma\hat{j}+\alpha\hat{k})=\lambda$, then value of tetrahedron (in cubic units) (if β, y, α, λ each > 0) is
(a) $\dfrac{1}{6}\dfrac{\lambda^3}{(\alpha\beta y)^3}$ (b) $\dfrac{\lambda^3}{\alpha\beta y}$
(c) $\dfrac{\lambda^3}{3\alpha\beta y}$ (d) $\dfrac{\lambda^3}{6\alpha\beta\gamma}$

7. Let the line $\dfrac{x-2}{3}=\dfrac{y-1}{-5}=\dfrac{z+2}{2}$ lie in the plane $x + 3y - \alpha z + \beta = 0$, then (α, β) equals
(a) (–6, 7) (b) (–5, 5) (c) (6, –7) (d) (–6, –7)

8. The distance of the point (1, –2, 3) from the plane $x - y + z - 5 = 0$ measured parallel to the line $\dfrac{x}{2}=\dfrac{y}{3}=\dfrac{z}{-6}$ is
(a) 4 (b) 3 (c) 2 (d) 1

9. The equation of the plane passing through the point (1, –1, 2) and (2, –2, 2) and is perpendicular to the plane $6x - 2y + 2z = 9$ is
(a) $x + y + 2z + 4 = 0$ (b) $x + y + 2z - 4 = 0$
(c) $x + y - 2z + 4 = 0$ (d) none of these

More than One Correct Answer Type

10. If $\vec{a},\vec{b},\vec{c}$ are three non-zero vectors, then which of the following statement(s) is/are true?
(a) $\vec{a}\times(\vec{b}\times\vec{c})+\vec{b}\times(\vec{c}\times\vec{a})+\vec{c}(\vec{a}\times\vec{b})=\vec{0}$
(b) If $\vec{a}+\vec{b}+\vec{c}=\vec{0}$ then $\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}<0$
(c) $\dfrac{(\vec{c}\times\vec{a})\cdot(\vec{b}\times\vec{c})}{(\vec{b}\times\vec{c})\cdot(\vec{a}\times\vec{b})}=1$ if $\vec{a}+\vec{b}+\vec{c}=0$
(d) none of these

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11. If $\vec{a}$ and $\vec{b}$ are any two unit vectors then which of the following statement(s) in the range of $\frac{5}{2}|\vec{a}+\vec{b}|+6|\vec{a}-\vec{b}|$ is/ are true?
(a) The maximum integer of the range is 13
(b) The minimum integer of the range is 5
(c) The most middle integer of the range is 9
(d) The average value of the maximum and minimum integer of the range is 9

12. A vector of magnitude 6 along the bisector of the angle between the two vectors $4\hat{i}-4\hat{j}+2\hat{k}$ and $2\hat{i}+4\hat{j}-4\hat{k}$ is
(a) $\frac{3}{\sqrt{26}}(\hat{i}-4\hat{j}-3\hat{k})$ (b) $\frac{6}{\sqrt{10}}(3\hat{i}-\hat{k})$
(c) $\frac{6}{\sqrt{26}}(\hat{i}-4\hat{j}+3\hat{k})$ (d) $\frac{6}{\sqrt{5}}(\hat{i}-3\hat{j})$

13. Let $A(1, -2, 3)$, $B(1, 1, 1)$, $C(1, 2, -3)$, $D(-1, -1, 1)$ be four points and $3x - y + 4z - 5 = 0$ be the equation of plane then which of the following line segments are intersected by the plane?
(a) AC (b) BC (c) BD (d) AD

14. Let $\vec{r}=2\hat{i}-2\hat{j}+3\hat{k}+\lambda(\hat{i}-\hat{j}+4\hat{k})$ be a line and equation of plane is $\vec{r}.(\hat{i}+5\hat{j}+\hat{k})=5$. Then,
(a) line is parallel to the plane
(b) the distance between line and plane $\frac{\sqrt{10}}{3\sqrt{3}}$
(c) the $\perp$ distance between line and plane is $\frac{\sqrt{10}}{3\sqrt{3}}$ units
(d) the point $(2, -2, 3)$ lies on the plane

15. Let $A(2, 3, 2)$, $B(1, 1, 1)$, $C(3, -2, 1)$, $D(7, 1, 4)$. Then,
(a) Points A, B, C, D are non coplanar
(b) Area of ΔBCD is $\frac{21}{2}$ sq. units
(c) Equation of the plane BCD is $3x + 2y - 6z + 1 = 0$
(d) Volume of tetrahedron $ABCD$ is $\frac{1}{2}$ cubic units

16. Consider the equations
$\vec{r}=(\hat{i}+2\hat{j}+3\hat{k})+\lambda(2\hat{i}+3\hat{j}+4\hat{k})$ ...(i)
$\vec{r}=(2\hat{i}+3\hat{j}+4\hat{k})+\mu(3\hat{i}+4\hat{j}+5\hat{k})$ ...(ii)
then which of the following statement(s) is/are true?
(a) $\vec{r}\cdot(4\hat{i}-3\hat{j})+1=0$ is the plane through the line (ii) and $\vec{r}\cdot(2\hat{i}-\hat{k})+1=0$ and is plane through the line (i)
(b) $\vec{r}\cdot(4\hat{j}-3\hat{k})+1=0$ is plane through the line (i) and $\vec{r}\cdot(5\hat{j}-4\hat{k})+1=0$ is plane through the line (ii)
(c) Lines (i) and (ii) are coplanar
(d) Lines (i) and (ii) intersecting

Comprehension Type

Paragraph for Q. No. 17-19

Let a plane P_1 passes through the point (2, 3, 4) and is parallel to the plane P_2 whose equation is $4x + 4y - 2z - 6 = 0$.

17. The distance of the point $(1, -1, 3)$ from the plane P_1 in units is
(a) 5 (b) 4 (c) 3 (d) 2

18. The coordinates of the foot of the perpendicular drawn from the point (4, 5, 6) to the plane P_2 are
(a) (–2, 3 7) (b) (2, 3, 7)
(c) (2, –3, 7) (d) (2, 3, –7)

19. The distance between the planes P_1 and P_2 is
(a) 1 units (b) 2 units
(c) 3 units (d) 4 units

Paragraph for Q.No. 20-23

Three vectors are $\vec{a},\vec{b},\vec{c}$ forming a right handed system and $\vec{a}\times\vec{b}=\vec{c}$, $\vec{b}\times\vec{c}=\vec{d}$, $\vec{c}\times\vec{a}=\vec{b}$.

20. The value of $3\vec{a}\cdot(\vec{b}\times\vec{c})+4\vec{b}\cdot(\vec{c}\times\vec{a})+5\vec{c}\cdot(\vec{a}\times\vec{b})$ equals
(a) 12 (b) 50
(c) $\sqrt{50}$ (d) none of these

21. If the vectors $3\vec{a}-4\vec{b}+3\vec{c}, \vec{a}+2\vec{b}-\vec{c}, 2\vec{c}-y\vec{b}+2\vec{c}$ are coplanar then value of y equals.
(a) $\frac{3}{8}$ (b) $\frac{8}{3}$
(c) 0 (d) none of these

22. If $\vec{x}=\vec{a}-2\vec{b}+\vec{c}, \vec{y}=-\vec{a}+3\vec{b}+\vec{c}$, $\vec{z}=2\vec{a}-\vec{b}+3\vec{c}$ the unit vector normal to the vector $\vec{x}+\vec{y}$ and $\vec{x}+\vec{z}$ is
(a) $\frac{1}{\sqrt{145}}(10\vec{a}-6\vec{b}-3\vec{c})$
(b) $\frac{1}{\sqrt{145}}(-10\vec{a}+6\vec{b}-3\vec{c})$
(c) $\frac{1}{\sqrt{145}}(10\vec{a}+6\vec{b}-3\vec{c})$
(d) none of these

23. Let $\vec{x}=2\vec{a}+\vec{b}$ and $y=\vec{a}-2\vec{b}$ then the point of intersection of the straight lines $\vec{r}\times\vec{x}=\vec{y}\times\vec{x}$ and $\vec{r}\times\vec{y}=\vec{x}\times\vec{y}$ is

(a) $\vec{a}+\vec{b}$ (b) $\vec{a}-\vec{b}$

(c) $3\vec{a}+2\vec{b}$ (d) $3\vec{a}-\vec{b}$

Matrix Match Type

24. Match the following.

	Column-I		Column-II
A.	$A(0, 3, 5)$ $B(3, 6, 5)$ $C(\alpha, 4, \beta)$ are vertices of triangle ABC such that the median through B is equally inclined to the co-ordinate axes then the value of $(9\alpha+\beta)$ is	p.	5
B.	Let $\vec{a}=\hat{i}+2\hat{j}-\hat{k}$ and $\vec{b}=\hat{i}-\hat{j}+2\hat{k}$ and $\vec{c}=\hat{i}+(x-3)\hat{j}-2x\hat{k}$. If the vector $\vec{c}$ lies in the plane of $\vec{a}$ and $\vec{b}$ then x^2 equals	q.	21
C.	Vectors $\vec{a},\vec{b},\vec{c}$ satisfies the condition $\vec{a}+\vec{b}+\vec{c}=\vec{0}$ such that $\lvert\vec{a}\rvert=1$, $\lvert\vec{b}\rvert=4$, $\lvert\vec{c}\rvert=2$ and $\mu=\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}$ then $\lvert 2\mu\rvert$ equals	r.	16
D.	If $\vec{a},\vec{b},\vec{c}$ be three vectors such that $\lvert\vec{a}\rvert=3$, $\lvert\vec{b}\rvert=4$, $\lvert\vec{c}\rvert=5$ and each vector is perpendicular to the sum of the other two vectors then the $\left\lvert\dfrac{\vec{a}+\vec{b}+\vec{c}}{\sqrt{2}}\right\rvert$ equals	s.	9

25. Match the following.

	Column-I		Column-II
A.	If the co-ordinates of the points A (3, 4, 6) and B(8, 6, 9) then projections of the line AB to the co-ordinate axis are	p.	0, 8, –8
B.	The equation of a line is $\vec{r}=3\hat{i}+4\hat{j}+5\hat{k}+\lambda(2\hat{i}+5\hat{j}+\hat{k})$ then d.r.'s of the line are	q.	5, 2, 3
C.	Let $\vec{a}=3\hat{i}+2\hat{j}+2\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}+2\hat{k}$ then d.r.'s of $(\vec{a}+\vec{b})\times(\vec{a}-\vec{b})$ are	r.	8, 8, 0
D.	A line passes through $A(-3, -2, 4)$ and $B(5, 6, 4)$ then d.r. of the line AB are	s.	2, 5, 3

Integer Type

26. If $\vec{a},\vec{b}$ and $\vec{c}$ are unit vectors perpendicular to each other and $\lvert\vec{b}\rvert=3$, $\lvert\vec{c}\rvert=4$ and $\vec{a}\times\vec{b}=\vec{c}$ then, the least value of $3\lvert\vec{a}-\vec{b}\rvert$ is

27. Let $\vec{a}$ is vector whose magnitude is unity and coplanar with the vectors $\vec{b}$ and $\vec{c}$, where $\vec{b}=\hat{i}-\hat{j}+3\hat{k}$ and $\vec{c}=3\hat{i}-\hat{j}+\hat{k}$ such that $\vec{a}$ is perpendicular to $\vec{b}$. If λ is the projection of $\vec{a}$ along $\vec{c}$ then the value of $\dfrac{11\sqrt{2}\lambda}{4}$ is

28. Let $A(1, 2, 3)$, $B(-1, -2, -1)$, $C(2, 3, 2)$ are three vertices of a parallelogram $ABCD$ if coordinates of $D(x, y, z)$ then $\dfrac{x+y+z}{17}$ is

29. The value of the tetrahedron which is included between the $4x-5y+6z=120$ and the co-ordinate planes is $600\,\lambda$ where λ is

30. Let $A(1, 2, -3)$ and $B(3, -4, 2)$ then the moment about the point $C(-2, 4, -6)$ of the force represented in magnitude and position by $\overline{AB}$ is $\dfrac{\sqrt{341}}{\lambda}$. Then value of λ is

SOLUTIONS

1. (d) : Let the plane $\vec{r}\cdot(\hat{i}-2\hat{j}+3\hat{k})=17$ divides the line joining the points $-2\hat{i}+4\hat{j}+7\hat{k}$ and $3\hat{i}-5\hat{j}+8\hat{k}$ in the ratio $\lambda:1$

$$(-2, 4, 7)\xrightarrow[\lambda\,:\,1]{R(x,\,y,\,z)}(3, -5, 8)$$

$$\therefore\ R(x,y,z)=\left(\frac{3\lambda-2}{\lambda+1},\frac{-5\lambda+4}{\lambda+1},\frac{8\lambda+7}{\lambda+1}\right)$$

Using $R(x, y, z)$ in the equation of the plane, we get

$$\left(\frac{3\lambda-2}{\lambda+1}\hat{i}+\frac{4-5\lambda}{\lambda+1}\hat{j}+\frac{8\lambda+7}{\lambda+1}\hat{k}\right)\cdot(\hat{i}-2\hat{j}+3\hat{k})=17$$

$\Rightarrow$ $(3\lambda-2)(1)+(4-5\lambda)(-2)+(8\lambda+7)(3)-17(\lambda+1)=0$

$\Rightarrow$ $20\lambda=6$ $\Rightarrow$ $\lambda=\dfrac{3}{10}$

$\therefore$ Required ratio is 3 : 10

2. **(b) :** As the given plane passes through $\vec{c}$ and is parallel to the vector $\vec{b}-\vec{c}$ and $\vec{a}$ so it is normal to $(\vec{b}-\vec{c})\times\vec{a}$. Hence its equation is given as

$$(\vec{r}-\vec{c})\cdot[(\vec{b}-\vec{c})\times\vec{a}]=0 \qquad \text{...(i)}$$

$\therefore$ Length of perpendicular from origin $O(0, 0, 0)$ to (i) is given by

$$\frac{|\vec{0}\cdot(\vec{b}\times\vec{a}+\vec{a}\times\vec{c})-[\vec{c}\ \vec{b}\ \vec{a}]|}{|\vec{b}\times\vec{a}+\vec{a}\times\vec{c}\,|}=\frac{1}{|\vec{b}\times\vec{a}+\vec{a}\times\vec{c}\,|}[\vec{c}\ \vec{b}\ \vec{a}]$$

3. **(c) :** Equation of line is given by

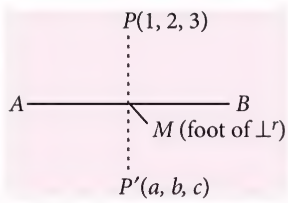
$\vec{r}=6\hat{i}+7\hat{j}+7\hat{k}+\lambda(3\hat{i}+2\hat{j}-2\hat{k})$

$\Rightarrow \dfrac{x-6}{3}=\dfrac{y-7}{2}=\dfrac{z-7}{-2}=\lambda$ (say)

$\therefore$ Coordinates of M are $(3\lambda+6, 2\lambda+7, -2\lambda+7)$

$\therefore$ d.r.'s of PM are $(3\lambda+5, 2\lambda+5, -2\lambda+4)$

and d.r.'s of AB are $(3, 2, -2)$



Now, $PM\perp AB$

$\Rightarrow 3(3\lambda+5)+2(2\lambda+5)-2(-2\lambda+4)=0$

$\Rightarrow 17\lambda=-17 \;\Rightarrow\; \lambda=-1$

$\therefore$ Foot of the perpendiculars, $M\cong(3, 5, 9)$

$\therefore$ Position vector of M is $3\hat{i}+5\hat{j}+9\hat{k}$

4. **(d) :** Translating the axes through $A(2, 3, 4)$ then, new position of A is $(0, 0, 0)$ and B is $(9, 6, 4)$

$\therefore$ Coterminous edges are of lengths 9, 6, 4 respectively

$\therefore$ Volume of parallelepiped $=9\times6\times4=216$ cu. units

5. **(b) :** As the line passes through $P(2, 3, 4)$ and has d.r.'s $(4, 5, 6)$

$\therefore$ Equation of line $=(2\hat{i}+3\hat{j}+4\hat{k})+\lambda(4\hat{i}+5\hat{j}+6\hat{k})$

$\therefore$ Any point on the line is $(2+4\lambda, 3+5\lambda, 4+6\lambda), \lambda\in R$

As the line meet the plane therefore any point that lie on the line satisfies the equation of the plane

$\therefore (2+4\lambda)+2(3+5\lambda)-3(4+6\lambda)+16=0$

$\Rightarrow -4\lambda=-12 \quad \therefore \lambda=3$

$\therefore$ Coordinates of $Q=(14, 18, 22)$

$\therefore PQ=\sqrt{(14-2)^2+(18-3)^2+(22-4)^2}$

$=\sqrt{144+225+324}=\sqrt{693}$ units

6. **(d) :** Cartesian equations of the planes are

$\Rightarrow \beta x+\gamma y=0$ ...(i); $\gamma y+\alpha z=0$ ...(ii);

$\alpha z+\beta x=0$ ...(iii) and $\beta x+\gamma y+\alpha z=\lambda$ ...(iv)

Now, plane (iv) cuts the plane (i) at $\left(0, 0, \dfrac{\lambda}{\alpha}\right)$

the plane (iv) cuts the plane (ii) at $\left(\dfrac{\lambda}{\beta}, 0, 0\right)$ and

the plane (iv) cuts the plane (iii) at $\left(0, \dfrac{\lambda}{\gamma}, 0\right)$

thus the planes (i), (ii) and (iii) are coordinate planes intersecting at origin (0, 0, 0). Therefore the lengths of the coterminous edges of the tetrahedron are $\left(\dfrac{\lambda}{\beta}, \dfrac{\lambda}{\gamma}, \dfrac{\lambda}{\alpha}\right)$

$\therefore$ Volume of tetrahedron $=\dfrac{1}{6}\left(\dfrac{\lambda}{\beta}\cdot\dfrac{\lambda}{\gamma}\cdot\dfrac{\lambda}{\alpha}\right)=\dfrac{\lambda^3}{6\alpha\beta\gamma}$

7. **(a) :** Equation of line is $\dfrac{x-2}{3}=\dfrac{y-1}{-5}=\dfrac{z+2}{2}$ and plane is $x+3y-\alpha z+\beta=0$.

The d.r.'s of the line are $(3, -5, 2)$

As the line lies in the plane, so every point of line satisfies the equation on plane and normal to plane must be $\perp$ to the line

$\therefore 3(1)+(-5)(3)+2(-\alpha)=0\Rightarrow\alpha=-6$

and $2+3(1)+2\alpha+\beta=0$

$\Rightarrow 2+3+2(-6)+\beta=0 \quad\therefore\quad \beta=7$

Thus $(\alpha, \beta)=(-6, 7)$

8. **(d)**

9. **(c) :** Equation of the plane through $(1, -1, 2)$ will be $a(x-1)+b(y+1)+c(z-2)=0$ which passes through $(2, -2, 2)$

$\therefore a(2-1)+b(-2+1)+c(2-2)=0$

$a-b=0 \;\Rightarrow\; a=b$

Also, the plane $a(x-1)+b(y+1)+c(z-2)=0$ is $\perp$ to $6x-2y+2z=9$

$\therefore 6a-2b+2c=0 \;\Rightarrow\; 3a-b+c=0$

$\Rightarrow c=-2a \qquad (\because a=b)$

$\therefore a(x-1)+a(y+1)-2a(z-2)=0$

$\Rightarrow x-1+y+1-2(z-2)=0 \Rightarrow x+y-2z+4=0$

10. **(a,b,c) :** (a) We know that

$$\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c} \qquad \text{...(i)}$$

$$\vec{b}\times(\vec{c}\times\vec{a})=(\vec{b}\cdot\vec{a})\vec{c}-(\vec{b}\cdot\vec{c})\vec{a} \qquad \text{...(ii)}$$

$$\vec{c}\times(\vec{a}\times\vec{b})=(\vec{c}\cdot\vec{b})\vec{a}-(\vec{c}\cdot\vec{a})\vec{b} \qquad \text{...(iii)}$$

Adding (i), (ii) and (iii) we get

$\vec{a}\times(\vec{b}\times\vec{c})+\vec{b}\times(\vec{c}\times\vec{a})+\vec{c}\times(\vec{a}\times\vec{b})=0$

(b) $\because \vec{a}+\vec{b}+\vec{c}=0 \Rightarrow (\vec{a}+\vec{b}+\vec{c})^2=0$

$\Rightarrow |\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})=$

$\Rightarrow |\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2=-2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})$

$\Rightarrow \vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}<0$ ($\because$ L.H.S. is a positive quantity)

(c) $\because\ \vec{a}+\vec{b}+\vec{c}=0$

$\Rightarrow\quad (\vec{a}+\vec{b})=-\vec{c} \Rightarrow \vec{c}\times(\vec{a}+\vec{b})=\vec{0} \Rightarrow \vec{c}\times\vec{a}=\vec{b}\times\vec{c}$ ...(i)

Again, $(\vec{a}+\vec{b})=-\vec{c} \quad\Rightarrow\quad \vec{b}\times(\vec{a}+\vec{b})=-\vec{b}\times\vec{c}$

$\Rightarrow\ \vec{a}\times\vec{b}=\vec{b}\times\vec{c}$ ...(ii)

By (i) and (ii) we get $\dfrac{(\vec{c}\times\vec{a})\cdot(\vec{b}\times\vec{c})}{(\vec{b}\times\vec{c})\cdot(a\times b)}=1$

11. (a,b,c,d) : $\because\ \vec{a},\vec{b}$ are unit vectors

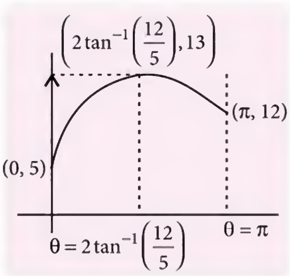
$\therefore\ |\vec{a}|=|\vec{b}|=1$, let θ be the angle between vectors $\vec{a}$ and $\vec{b}$.

Now $|\vec{a}+\vec{b}|^2=|\vec{a}|^2+|\vec{b}|^2+2\vec{a}\cdot\vec{b} \Rightarrow |\vec{a}+\vec{b}|=2\cos\dfrac{\theta}{2}$

and $|\vec{a}-\vec{b}|^2=|\vec{a}|^2+|\vec{b}|^2-2\vec{a}\cdot\vec{b} \Rightarrow |\vec{a}-\vec{b}|=2\sin\dfrac{\theta}{2}$

Let $f(\theta)=\dfrac{5}{2}\left(2\cos\dfrac{\theta}{2}\right)+6\left(2\sin\dfrac{\theta}{2}\right),\ \theta\in[0,\pi]$

$\therefore\ f'(\theta)=\dfrac{-5}{2}\sin\dfrac{\theta}{2}+6\cos\dfrac{\theta}{2}$



$f'(\theta)=0 \Rightarrow \tan\dfrac{\theta}{2}=\dfrac{12}{5}$

$\therefore\quad \sin\dfrac{\theta}{2}=\dfrac{12}{13},\ \cos\dfrac{\theta}{2}=\dfrac{5}{13}$

$\therefore\quad f(0)=5(1)+12(0)=5$

$$f\left(\theta=2\tan^{-1}\left(\frac{12}{5}\right)\right)=\frac{5(5)}{13}+12\left(\frac{12}{13}\right)=13$$

and $f(\theta=\pi)=5(0)+12(1)=12 \quad\therefore\quad$ Range $=[5, 13]$

$\therefore$ Maximum integral value of range = 13

Minimum integral value of range = 5

Most middle integral value of range 5, 6, 7, 8, 9, 10, 11, 12, 13 is 9.

Average of minimum integral value and maxima integral value of the range is $\dfrac{5+13}{2}=9$

12. (b,d) : Let $\vec{a}=4\hat{i}-4\hat{j}+2\hat{k},\ \ \vec{b}=2\hat{i}+4\hat{j}-4\hat{k}$

$\therefore\quad |\vec{a}|=|\vec{b}|=6$

$\therefore$ Vectors along bisectors is

$$\frac{\vec{a}}{|\vec{a}|}\pm\frac{\vec{b}}{|\vec{b}|}=\frac{1}{6}[(4\hat{i}-4\hat{j}+2\hat{k})\pm(2\hat{i}+4\hat{j}-4\hat{k})]$$

$\therefore$ Vectors along bisectors are

$$=\frac{1}{6}[(4\hat{i}-4\hat{j}+2\hat{k}+2\hat{i}+4\hat{j}-4\hat{k}),(4\hat{i}-4\hat{j}+2\hat{k}-2\hat{i}-4\hat{j}+4\hat{k})]$$

$$=\frac{1}{6}[(6\hat{i}-2\hat{k}),(2\hat{i}-8\hat{j}+6\hat{k})]=\left(\hat{i}-\frac{1}{3}\hat{k},\frac{\hat{i}}{3}-\frac{4}{3}\hat{j}+\hat{k}\right)$$

$\therefore$ Required vectors are

$$6\frac{\left(\hat{i}-\frac{1}{3}\hat{k}\right)}{\sqrt{\frac{10}{9}}},6\frac{\frac{\hat{i}}{3}-\frac{4}{3}\hat{j}+\hat{k}}{\sqrt{\left(\frac{1}{3}\right)^2+\left(\frac{-4}{3}\right)^2+1^2}}$$

$$=6\left(\frac{3\hat{i}-\hat{k}}{\sqrt{10}}\right),6\cdot\left(\frac{\hat{i}-4\hat{j}+3\hat{k}}{\sqrt{26}}\right)$$

or $\dfrac{6}{\sqrt{10}}(3\hat{i}-\hat{k}),\dfrac{6}{\sqrt{26}}(\hat{i}-4\hat{j}+3\hat{k})$

13. (a, b, c, d) : For point $A(1, -2, 3)$ and the plane $3x-y+4z-5=0$ we have $3(1)+2+4(3)-5>0$

For point $B(1, 1, 1)$ $3x-y+4z-5=3-1+4-5>0$

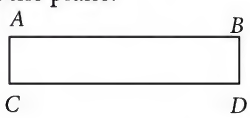
For point $C(1, 2, -3)$,

$3x-y+4z-5=3-2-12-5<0$

For point $D(-1, -1, 1)$,

$3x-y+4z-5=-3+1+4-5<0$

$\therefore$ Points A, B are on one side of the plane and C, D on the other side of the plane.



$\therefore$ The line segments AC, AD, BC, BD intersect the plane.

14. (a,c)

15. (a, b, c, d) : Equation of the plane through the points A, B, C is

(a) $\begin{vmatrix} x-2 & y-3 & z-2 \\ -1 & -2 & -1 \\ 1 & -5 & -1 \end{vmatrix}=0$

$\Rightarrow\quad (x-2)(-3)-(y-3)(2)+(z-2)(7)=0$ ...(i)

Using $D(7, 1, 4)$ in (i), we have

$(7-2)(-3)-(1-3)(2)+(4-2)(7)\neq 0$

$\therefore$ Points A, B, C, D are non-coplanar .

(b) Area of ΔBCD : $B(1, 1, 1)$, $C(3, -2, 1)$, $D(7, 1, 4)$
$=B(x_1, y_1, z_1)$, $C(x_2, y_2, z_2)$, $D(x_3, y_3, z_3)$

$\therefore$ Area of $\Delta BCD=\Delta=\sqrt{\Delta_x^2+\Delta_y^2+\Delta_z^2}$

where $\Delta_x=\dfrac{1}{2}\begin{vmatrix} y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \\ 1 & 1 & 1 \end{vmatrix}$ and so on

$$\therefore\quad \Delta_x=\frac{1}{2}\begin{vmatrix} 1 & -2 & 1 \\ 1 & 1 & 4 \\ 1 & 1 & 1 \end{vmatrix},\Delta_y=\frac{1}{2}\begin{vmatrix} 1 & 3 & 7 \\ 1 & 1 & 4 \\ 1 & 1 & 1 \end{vmatrix},\ \Delta_z=\frac{1}{2}\begin{vmatrix} 1 & 3 & 7 \\ 1 & -2 & 1 \\ 1 & 1 & 1 \end{vmatrix}$$

$\therefore \quad \Delta_x = \frac{9}{2},\ \Delta_y = 3,\ \Delta_z = 9$

$\therefore \quad$ Area of $\Delta BCD = \Delta = \sqrt{\frac{81}{4}+9+81} = \frac{21}{2}$ sq. units

(c) Equation of the plane $B(1, 1, 1), C(3, -2, 1), D(7, 1, 4)$ is

$$\begin{vmatrix} x-1 & y-1 & z-1 \\ 2 & -3 & 0 \\ 6 & 0 & 3 \end{vmatrix} = 0$$

$\Rightarrow \quad (x-1)(-9) - (y-1)(6) + (z-1)(18) = 0$

$\Rightarrow \quad 3x + 2y - 6z + 1 = 0$

(d) Volume of tetrahedron $ABCD$

$= \frac{1}{3} \times$ area of ΔBCD

$\times \perp$ distance from A to the plane of BCD

$= \frac{1}{3} \times \frac{21}{2} \times \frac{1}{7} = \frac{1}{2}$ cubic units

16. (a,b,c,d) : As $\vec{r} = (\hat{i}+2\hat{j}+3\hat{k}) + \lambda(2\hat{i}+3\hat{j}+4\hat{k})$

$\Rightarrow \quad \frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} = \lambda \qquad$ ...(i)

and $\vec{r} = (2\hat{i}+3\hat{j}+4\hat{k}) + \mu(3\hat{i}+4\hat{j}+5\hat{k})$

$\Rightarrow \quad \frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5} \qquad$ ...(ii)

$\therefore \quad$ Equation of the planes through (i) are

$\frac{x-1}{2} = \frac{y-2}{3},\ \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-1}{2} = \frac{z-3}{4}$

$\Rightarrow \quad 3x - 2y + 1 = 0,\ 4y - 3z + 1 = 0$ and $2x - z + 1 = 0$

$\Rightarrow \quad \vec{r}\cdot(3\hat{i}-2\hat{j})+1=0,\ \vec{r}\cdot(4\hat{j}-3\hat{k})+1=0$

and $\vec{r}\cdot(2\hat{i}-\hat{k})+1=0$

Again, equation of planes through line (ii) are

$\frac{x-2}{3} = \frac{y-3}{4},\ \frac{y-3}{4} = \frac{z-4}{5}$ and $\frac{x-2}{3} = \frac{z-4}{5}$

$\Rightarrow \quad 4x - 3y + 1 = 0,\ 5y - 4z + 1 = 0$ and $5x - 3z + 2 = 0$

$\Rightarrow \quad \vec{r}\cdot(4\hat{i}-3\hat{j})+1=0,\ \vec{r}\cdot(5\hat{j}-4\hat{k})+1=0$

and $\vec{r}\cdot(5\hat{i}-3\hat{k})+2=0$

Again any point on the line (1) is $P(2\lambda + 1, 3\lambda + 2, 4\lambda + 3)$ and $4x - 3y + 1 = 0, 5x - 3z + 2 = 0, 5y - 4z + 1 = 0$ or $5(2\lambda + 1) - 3(4\lambda + 3) + 2 = 0$ each gives the same value of $\lambda = -1$. So the lines (i) and (ii)are coplanar. Also, the lines are coplanar therefor they will intersect and it can be verified by using the formula,

$$d = \text{Shortest distance} = \left|\frac{(\vec{a}_2-\vec{a}_1)\cdot(\vec{b}_1\times\vec{b}_2)}{\vec{b}_1\times\vec{b}_2}\right|$$

where $\vec{a}_2 - \vec{a}_1 = \hat{i}+\hat{j}+\hat{k}$

Now, $\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = -\hat{i}+2\hat{j}-\hat{k}$

$\therefore \quad (\vec{a}_2-\vec{a}_1)\cdot(\vec{b}_1\times\vec{b}_2) = (\hat{i}+\hat{j}+\hat{k})\cdot(-\hat{i}+2\hat{j}-\hat{k}) = 0$

$\therefore \quad d = 0.$

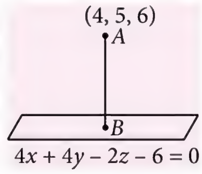
17. (c) : Equation of the plane P_1 passing through the point (2, 3, 4) and parallel to plane $4x + 4y - 2z - 6 = 0$ is given by $4(x-2) + 4(y-3) - 2(z-4) = 0$

$\Rightarrow \quad 4x + 4y - 2z - 12 = 0 \qquad$ ...(i)

$\therefore \quad$ Distance from $(1, -1, 3)$ to the plane (i)

$$= \left|\frac{4(1)+4(-1)-2(3)-12}{\sqrt{4^2+4^2+(-2)^2}}\right| = \left|\frac{18}{6}\right| = 3 \text{ units}$$

18. (b) : Equation of the line AB having d.r.'s (4, 4, –2) through the point (4, 5, 6) is



$\frac{x-4}{4} = \frac{y-5}{4} = \frac{z-6}{-2} = \lambda$ (say)

$\Rightarrow \quad x - 4 = 4\lambda,\ y - 5 = 4\lambda,\ z - 6 = -2\lambda$

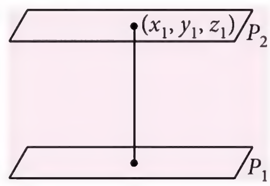
$\Rightarrow \quad x = 4\lambda + 4,\ y = 4\lambda + 5,\ z = -2\lambda + 6$

Using this point on the plane P_2, we have

$4(4\lambda + 4) + 4(4\lambda + 5) - 2(-2\lambda + 6) - 6 = 0$

$\Rightarrow \quad 16\lambda + 16\lambda + 4\lambda = -16 - 20 + 12 + 6$

$\Rightarrow \quad 36\lambda = -18 \quad \Rightarrow \quad \lambda = \frac{-1}{2}$



and foot of the $\perp$ are

$x = 4(-1/2) + 4,$

$y = 4(-1/2) + 5,$

$z = -2(-1/2) + 6$

$x = 2, y = 3, z = 7 \quad \therefore \quad (x, y, z) = (2, 3, 7)$

19. (a) : Given $P_1 : 4x + 4y - 2z - 12 = 0$

$P_2 : 4x + 4y - 2z - 6 = 0$

Let (x_1, y_1, z_1) be any point on P_2

$\therefore \quad 4x_1 + 4y_1 - 2z_1 - 6 = 0 \qquad$ ...(*)

Now distance from, (x_1, y_1, z_1) to plane P_1

$\therefore \quad$ Required distance

$$= \frac{|4x_1+4y_1-2z-6-6|}{\sqrt{4^2+4^2+(-2)^2}} = \frac{6}{6} = 1 \text{ unit}$$

20. (a) : $\because \quad \vec{a}, \vec{b}, \vec{c}$ forming a right handed system

$\therefore \quad \vec{a}\cdot(\vec{b}\times\vec{c}) = \vec{b}\cdot(\vec{c}\times\vec{a}) = \vec{c}\cdot(\vec{a}\times\vec{b}) = [\vec{a}\ \vec{b}\ \vec{c}]$

and $\vec{a}\times\vec{b} = \vec{c} \Rightarrow \vec{c}\cdot(\vec{a}\times\vec{b}) = \vec{c}\cdot\vec{c} \Rightarrow [\vec{a}\,\vec{b}\,\vec{c}] = 1$

$= (3+4+5)\,\vec{a}\cdot(\vec{b}\times\vec{c}) = 12[\vec{a}\ \vec{b}\ \vec{c}] = 12$

21. (b) : $\because \quad 3\vec{a} - 4\vec{b} + 3\vec{c},\ \vec{a} + 2\vec{b} - \vec{c},$

$2\vec{a} - y\vec{b} + 2\vec{c}$ are coplanar

$\therefore \begin{vmatrix} 3 & -4 & 3 \\ 1 & 2 & -1 \\ 2 & -y & 2 \end{vmatrix} = 0$

$\Rightarrow 3(4 - y) + 4(4) + 3(-y - 4) = 0$

$\Rightarrow -6y = -16 \Rightarrow y = \dfrac{8}{3}$

22. (c) : $\vec{x} = \vec{a} - 2\vec{b} + \vec{c},\ \vec{y} = -\vec{a} + 3\vec{b} + \vec{c},$

$\vec{z} = 2\vec{a} - \vec{b} + 3\vec{c}$

So, $\vec{x} + \vec{y} = \vec{b} + 2\vec{c},\ \vec{x} + \vec{z} = 3\vec{a} - 3\vec{b} + 4\vec{c}$

$\therefore (\vec{x} + \vec{y}) \times \vec{x} + \vec{z}) = \begin{vmatrix} \vec{a} & \vec{b} & \vec{c} \\ 0 & 1 & 2 \\ 3 & -3 & 4 \end{vmatrix}$

$\therefore \vec{a}(10) + 6\vec{b} + \vec{c}(-3) = 10\vec{a} + 6\vec{b} - 3\vec{c}$

$\therefore$ Unit vector in the direction of

$(\vec{x} + \vec{y}) \times (\vec{x} + \vec{z}) = \dfrac{10\vec{a} + 6\vec{b} - 3\vec{c}}{\sqrt{145}}$

23. (d) : Given $\vec{x} = 2\vec{a} + \vec{b},\ \vec{y} = \vec{a} - 2\vec{b}$

Now, $\vec{r} \times \vec{x} = \vec{y} \times \vec{x} = (\vec{r} - \vec{y}) \times \vec{x} = 0$

$\Rightarrow \vec{r} - \vec{y} \parallel \vec{x}$

$\Rightarrow \vec{r} - \vec{y} = \lambda\vec{x} \Rightarrow \vec{r} = \vec{y} + \lambda\vec{x}$ ...(i)

and $\vec{r} \times \vec{y} = \vec{x} \times \vec{y} \Rightarrow (\vec{r} - \vec{x}) \times \vec{y} = 0 \Rightarrow \vec{r} - \vec{x} \parallel \vec{y}$

$\Rightarrow \vec{r} - \vec{x} = \mu\vec{y} \Rightarrow \vec{r} = \vec{x} + \mu\vec{y}$ ...(ii)

From (i) and (ii) we have

$\vec{y} + \lambda\vec{x} = \vec{x} + \mu\vec{y}$

$\Rightarrow (\vec{a} - 2\vec{b}) + \lambda(2\vec{a} + \vec{b}) = (2\vec{a} + \vec{b}) + \mu(\vec{a} - 2\vec{b})$

$\Rightarrow \vec{a}(1 + 2\lambda) + \vec{b}(-2 + \lambda) = \vec{a}(2 + \mu) + \vec{b}(1 - 2\mu)$

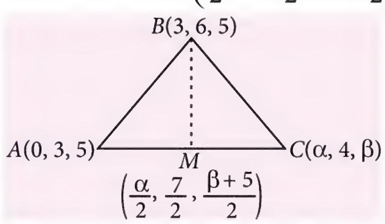
$\Rightarrow 1 + 2\lambda = 2 + \mu$ and $-2 + \lambda = 1 - 2\mu \Rightarrow \lambda = \mu = 1$

$\therefore$ Point of intersection from (i) and (ii) is $3\vec{a} - \vec{b}$.

24. A $\to$ s, B $\to$ r, C $\to$ q, D $\to$ p

A. Let M is mid point of $AC = \dfrac{\alpha}{2}, \dfrac{7}{2}, \left(\dfrac{\beta + 5}{2}\right)$

d.r.'s of the median BM are $\left(\dfrac{\alpha}{2} - 3,\ \dfrac{7}{2} - 6,\ \dfrac{\beta + 5}{2} - 5\right)$



But the median is equally inclined to the axes

$\therefore$ Its d.c.'s are equal

$\therefore \dfrac{\alpha}{2} - 3 = \dfrac{7}{2} - 6 \Rightarrow \dfrac{\alpha}{2} = \dfrac{7}{2} - 3 \Rightarrow \alpha = 1$

and $\dfrac{\beta + 5}{2} - 5 = \dfrac{-5}{2} \Rightarrow \beta = 0$

$\therefore 9\alpha + \beta = 9(1) + 0 = 9$

B. As vectors $\vec{a}, \vec{b}, \vec{c}$ are coplanar

$\therefore \begin{vmatrix} 1 & 2 & -1 \\ 1 & -1 & 2 \\ 1 & x - 3 & -2x \end{vmatrix} = 0$

$\Rightarrow 1(2x - 2x + 6) - 2(-2x - 2) - 1(x - 2) = 0$

$\Rightarrow x = -4 \quad \therefore x^2 = 16$

C. Given $\vec{a} + \vec{b} + \vec{c} = \vec{0}$ and $\mu = \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}$

$\therefore (\vec{a} + \vec{b} + \vec{c}) \cdot (\vec{a} + \vec{b} + \vec{c}) = 0$

$\Rightarrow a^2 + b^2 + c^2 + 2\mu = 0 \Rightarrow 2\mu = -21 \therefore |2\mu| = 21$

D. $\because$ Each vector is $\perp$ to the sum of other two vector

$\therefore \vec{a} \cdot (\vec{b} + \vec{c}) = 0, \vec{b} \cdot (\vec{a} + \vec{c}) = 0, \vec{c} \cdot (\vec{a} + \vec{b}) = 0$ ...(*)

Now, $|\vec{a} + \vec{b} + \vec{c}|^2 = (\vec{a} + \vec{b} + \vec{c})(\vec{a} + \vec{b} + \vec{c})$

$= \vec{a} \cdot \vec{a} + \vec{b} \cdot \vec{b} + \vec{c} \cdot \vec{c} = a^2 + b^2 + c^2 = 50$

$\therefore |\vec{a} + \vec{b} + \vec{c}| = 5\sqrt{2} \Rightarrow \left|\dfrac{\vec{a} + \vec{b} + \vec{c}}{\sqrt{2}}\right| = 5$

25. A $\to$ q, B $\to$ s, C $\to$ p, D $\to$ r

A. Given $A(3, 4, 6)$ and $B(8, 6, 9)$ $\therefore x_2 - x_1 = 5$, $y_2 - y_1 = 2$, $z_2 - z_1 = 3$ and d.c.'s of x, y, z axes are (1, 0, 0), (0, 1, 0) and (0, 0, 1) respectively.

Also the projection of a line segment having d.r.'s $x_2 - x_1, y_2 - y_1, z_2 - z_1$ on a line whose d.c.'s are l, m, n is given by $(x_2 - x_1)\,l + (y_2 - y_1)\,m + (z_2 - z_1)n$

$\therefore$ The projections of the line AB on the co-ordinate x axis are

$(x_2 - x_1)l + (y_2 - y_1)m + (z_2 - z_1)n = 5(1) + 2(0) + 3(0) = 5$

Similarly projection of y and z axes are 2, 3 respectively

$\therefore$ Projection of line AB on axis are 5, 2, 3

B. We have, $\vec{r} = (3\hat{i} + 4\hat{j} + 5\hat{k}) + \lambda(2\hat{i} + 5\hat{j} + 3\hat{k})$

$\Rightarrow \vec{b} = 2\hat{i} + 5\hat{j} + 3\hat{k} \ \therefore$ d.r.'s $= 2, 5, 3$

C. As $\vec{a} = 3\hat{i} + 2\hat{j} + 2\hat{k},\ \vec{b} = \hat{i} + 2\hat{j} + 2\hat{k}$

$\therefore \vec{a} + \vec{b} = 4\hat{i} + 4\hat{j} + 4\hat{k}$ and $\vec{a} - \vec{b} = 2\hat{i} + 0\hat{j} + 0\hat{k}$

$\therefore (\vec{a} + \vec{b}) \times (\vec{a} - \vec{b}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 4 & 4 \\ 2 & 0 & 0 \end{vmatrix}$

$= \hat{i}(0) - \hat{j}(-8) + \hat{k}(-8) = 0\hat{i} + 8\hat{j} - 8\hat{k}$

$\therefore$ d.r.'s of $(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b})$ are $(0, 8, -8)$

D. We have, $A(-3, -2, 4)$, $B(5, 6, 4)$

$\therefore$ d.r.'s of $AB = x_2 - x_1, y_2 - y_1, z_2 - z_1$

$= 5 - (-3), 6 - (-2), 4 - 4 = 8, 8, 0$

26. (4) : Given $|\vec{b}|=3, |\vec{c}|=4 \because \vec{a}\times\vec{b}=\vec{c}$

$\Rightarrow |\vec{a}\times\vec{b}|=|\vec{c}| \Rightarrow |\vec{a}||\vec{b}|\sin\alpha=|\vec{c}|$

$\Rightarrow |\vec{a}|=\frac{4}{3}\operatorname{cosec}\alpha$ ...(*)

Now $|\vec{a}-\vec{b}|^2=|\vec{a}|^2+|\vec{b}|^2-2\vec{a}\cdot\vec{b}$

$=|\vec{a}|^2+9-2|\vec{a}||\vec{b}|\cos\alpha$

$=\frac{16}{9}\operatorname{cosec}^2\alpha+9-2\cdot4\operatorname{cosec}\alpha\cdot\cos\alpha$ (using (*))

$=\frac{16}{9}(1+\cot^2\alpha)+9-8\cot\alpha$

$=\frac{16}{9}+\left(\frac{16}{9}\cot^2\alpha-8\cot\alpha+9\right)$

$=\frac{16}{9}+\left(\frac{4}{3}\cot\alpha-3\right)^2\geq\frac{16}{9}$

$\therefore |\vec{a}-\vec{b}|^2\geq\frac{16}{9} \Rightarrow |\vec{a}-\vec{b}|\geq\frac{4}{3} \Rightarrow 3|\vec{a}-\vec{b}|\geq4$

$\Rightarrow$ least value of $3|\vec{a}-\vec{b}|=4$

27. (3) : Given $\vec{b}=\hat{i}-\hat{j}+3\hat{k},\ \vec{c}=3\hat{i}-\hat{j}+\hat{k}$

$\Rightarrow |\vec{b}|=|\vec{c}|=\sqrt{11}$

Let $\vec{a}=x\vec{b}+y\vec{c}$ where x and $y\in R$ and $|\vec{a}|=1$ ...(*)

$\therefore \vec{a}=x(\hat{i}-\hat{j}+3\hat{k})+y(3\hat{i}-\hat{j}+\hat{k})$

$\vec{a}=\hat{i}(x+3y)+\hat{j}(-x-y)+\hat{k}(3x+y)$

Now $\vec{a}\perp\vec{b} \therefore \vec{a}\cdot\vec{b}=0$

$\Rightarrow [\hat{i}(x+3y)+\hat{j}(-x-y)+\hat{k}(3x+y)]\cdot[\hat{i}-\hat{j}+3\hat{k}]=0$

$\Rightarrow x+3y+x+y+3(3x+y)=0$

$\Rightarrow 11x+7y=0 \Rightarrow y=\frac{-11}{7}x$

$\therefore \vec{a}=\hat{i}\left(x+\frac{3(-11x)}{7}\right)+\hat{j}\left(-x+\frac{11}{7}x\right)+\hat{k}\left(3x-\frac{11}{7}x\right)$

$=\hat{i}\left(\frac{-26}{7}x\right)+\hat{j}\left(\frac{4x}{7}\right)+\hat{k}\left(\frac{10x}{7}\right)$

$\Rightarrow |\vec{a}|=1 \Rightarrow \left(\frac{(26)^2}{49}+\frac{4^2}{49}+\frac{10^2}{49}\right)x^2=1$

$\Rightarrow x^2=\frac{49}{792} \quad \therefore x=\pm\frac{7}{\sqrt{792}}$ or $\pm\frac{7}{6\sqrt{22}}$

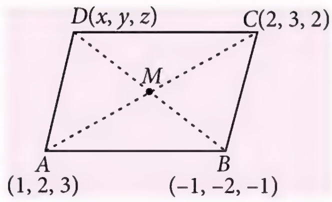
$\therefore \vec{a}=\pm\frac{7}{6\sqrt{22}}\left(-\frac{26}{7}\hat{i}+\frac{4}{7}\hat{j}+\frac{10}{7}\hat{k}\right)$

Now projection of $\vec{a}$ along $\vec{c}=\lambda=\frac{|\vec{a}\cdot\vec{c}|}{|\vec{c}|}$

$$=\left|\frac{\frac{7}{6\sqrt{22}}\left(\frac{-26}{7}\hat{i}+\frac{4}{7}\hat{j}+\frac{10}{7}\hat{k}\right)\cdot(3\hat{i}-\hat{j}+\hat{k})}{\sqrt{11}}\right|=\frac{72}{6\times11\sqrt{2}}$$

$\therefore \frac{11\sqrt{2}\lambda}{4}=3.$

28. (1) : $\because$ *ABCD* is a parallelogram diagonals *AC* and *BD* bisect each other *i.e.*, the mid point of the segment *AC* is same as the mid point of the segment *BD*



$\Rightarrow M\left(\frac{2+1}{2},\frac{2+3}{2},\frac{3+2}{2}\right)=\left(\frac{x-1}{2},\frac{y-2}{2},\frac{z-1}{2}\right)$

$\Rightarrow x-1=3, y-2=5, z-1=5$

$x=4, y=7, z=6$

$\therefore x+y+z=4+7+6=17$

29. (4) : Given equation of plane is $4x-5y+6z=120$ which meet the coordinate axes at $P(30, 0, 0)$, $Q(0, -24, 0)$ and $R(0, 0, 20)$ and the coordinate of the origin are $(0, 0, 0)$

$\therefore$ Volume of tetrahedron

$$=\left|\frac{1}{6}\begin{vmatrix}30&0&0\\0&-24&0\\0&0&20\end{vmatrix}\right|=\left|\frac{1}{6}\left(\frac{20\times24\times30}{-1}\right)\right|$$

$=2400$ cubic units

30. (1) : If 'O' is the origin then $\overrightarrow{OA}=\hat{i}+2\hat{j}-3\hat{k}$ and $\overrightarrow{OB}=3\hat{i}-4\hat{j}+2\hat{k}$ and $\overrightarrow{OC}=-2\hat{i}+4\hat{j}-6\hat{k}$

$\therefore \overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=(3\hat{i}-4\hat{j}+2\hat{k})-(\hat{i}+2\hat{j}-3\hat{k})$

$=2\hat{i}-6\hat{j}+5\hat{k}$

$\therefore \overrightarrow{CA}=\overrightarrow{OA}-\overrightarrow{OC}=(\hat{i}+2\hat{j}-3\hat{k})-(-2\hat{i}+4\hat{j}-6\hat{k})$

$=3\hat{i}-2\hat{j}+3\hat{k}$

$\therefore$ Required moment $=\overrightarrow{CA}\times\overrightarrow{AB}=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\3&-2&3\\2&-6&5\end{vmatrix}$

$=\hat{i}(-10+18)-\hat{j}(15-6)+\hat{k}(-18+4)=8\hat{i}-9\hat{j}-14\hat{k}$

$\therefore |\overrightarrow{CA}\times\overrightarrow{AB}|=|8\hat{i}-9\hat{j}-14\hat{k}|$

$=\sqrt{8^2+(-9)^2+(-14)^2}=\sqrt{341}$

MATHS MUSING

Maths Musing was started in January 2003 issue of Mathematics Today. The aim of Maths Musing is to augment the chances of bright students seeking admission into IITs with additional study material.

During the last 10 years there have been several changes in JEE pattern. To suit these changes Maths Musing also adopted the new pattern by changing the style of problems. Some of the Maths Musing problems have been adapted in JEE benefitting thousand of our readers. It is heartening that we receive solutions of Maths Musing problems from all over India.

Maths Musing has been receiving tremendous response from candidates preparing for JEE and teachers coaching them. We do hope that students will continue to use Maths Musing to boost up their ranks in JEE Main and Advanced.

PROBLEM Set 181

JEE MAIN

1. If $1, \omega, \omega^2, ..., \omega^{n-1}$ are, n^{th} roots of unity, the value of $(9 - \omega)(9 - \omega^2) ... (9 - \omega^{n-1})$ will be

   (a) n (b) 0 (c) $\frac{9^n - 1}{8}$ (d) $\frac{9^n + 1}{8}$

2. The numbers $3^{2\sin 2x - 1}$, 14, $3^{4 - 2\sin 2x}$ form first three terms of an A.P., its fifth term is equal to

   (a) -25 (b) -12 (c) 40 (d) 53

3. The value of $\sum_{i=0}^{n} \sum_{j=1}^{n} {}^nC_j \, {}^jC_i$, $i \le j$ is

   (a) $3^n - 1$ (b) 0
   (c) 2^n (d) none of these

4. If $At^4 + Bt^3 + Ct^2 + Dt + E = \begin{vmatrix} t^2+3t & t-1 & t-3 \\ t+1 & 2-t & t-3 \\ t-3 & t+4 & 3t \end{vmatrix}$, then E equals

   (a) 33 (b) -39 (c) 27 (d) 24

5. The rank of the matrix $\begin{pmatrix} -1 & 2 & 5 \\ 2 & -4 & a-4 \\ 1 & -2 & a+1 \end{pmatrix}$ (where $a = -6$) is

   (a) 1 (b) 2 (c) 3 (d) 4

JEE ADVANCED

6. Let $f: R \to R$, $g: R \to R$, be two given functions such that f is injective and g is surjective, then which of the following is injective?

   (a) gof (b) fog (c) gog (d) fof

COMPREHENSION

Let $f(x)$ be a continuous function defined on the closed interval $[a, b]$, then

$$\lim_{n\to\infty} \sum_{r=0}^{n-1} \frac{1}{n} f\left(\frac{r}{n}\right) = \int_0^1 f(x)dx$$

7. The value of $\lim_{n\to\infty} \left\{ \frac{1}{n} + \frac{n^2}{(n+1)^3} + \frac{n^2}{(n+2)^3} + ... + \frac{1}{8n} \right\}$ is

   (a) 5/4 (b) 3/4 (c) 5/8 (d) 3/8

8. The value of
$$\lim_{n\to\infty} \left\{ \tan\left(\frac{\pi}{2n}\right) \tan\left(\frac{2\pi}{2n}\right) \tan\left(\frac{3\pi}{2n}\right) ... \tan\left(\frac{n\pi}{2n}\right) \right\}^{1/n}$$ is

   (a) 1 (b) 2
   (c) 3 (d) not defined

INTEGER TYPE

9. If $\lim_{x\to 0} \frac{\sin(\sin x) - \sin x}{ax^3 + bx^5 + c} = -\frac{1}{12}$, then find the value of a.

MATRIX MATCH

10. Match the following.

	List-I		List-II
P.	$y = \sin^{-1}(3x - 4x^3)$, then $\frac{dy}{dx}$ is	1.	$\frac{3}{1+x^2}$, $x \in \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$
Q.	$y = \cos^{-1}(4x^3 - 3x)$, then $2\frac{dy}{dx}$ is	2.	$\frac{-8}{\ln 2} + \frac{32}{\pi^2 + 16}$
R.	$y = \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$, then $\frac{dy}{dx}$ is	3.	$-\frac{6}{\sqrt{1-x^2}}$, $x \in \left(-1, -\frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right)$
S.	If $f(x) = (\log_{\cos x} \sin x) \times (\log_{\sin x} \cos x)^{-1} + \sin^{-1}\left(\frac{2x}{1+x^2}\right)$, then $f'\left(\frac{\pi}{4}\right) =$	4.	$\frac{3}{\sqrt{1-x^2}}$, $x \in \left(-\frac{1}{2}, \frac{1}{2}\right)$

	P	Q	R	S
(a)	4	3	1	2
(b)	1	3	2	4
(c)	1	4	2	3
(d)	2	3	1	4

See Solution Set of Maths Musing 180 on page no 88

Exam Dates
OFFLINE : 8th April
ONLINE : 15th & 16th April

JEE Main 2018

MOCK TEST PAPER

Series-7

Time: 1 hr 15 min.

The entire syllabus of Mathematics of JEE MAIN is being divided into eight units, on each unit there will be a Mock Test Paper (MTP) which will be published in the subsequent issues. The syllabus for module break-up is given below:

Unit No.7	Topic	Syllabus In Detail
	Co-ordinate Geometry-3D	Direction ratios and direction cosines. Angle between two intersecting lines. Skew lines, the shortest distance between them and its equation. Equation of a line and a plane in different forms, intersection of a line and a plane, coplanar lines.
	Differential Calculus	Rolle's and Lagrange's Mean value theorems, Applications of derivatives: Rate of change of quantities, monotonic-increasing and decreasing functions, Maxima and minima of functions of one variable, tangents and normals.
	Integral Calculus	Integral as an anti-derivative. Fundamental integrals involving algebraic, trigonometric, exponential and logarithmic functions. Integration by substitution, by parts and by partial fractions. Integration using trigonometric identities. Evaluation of simple integrals of type: $\int\frac{dx}{x^2\pm a^2}, \int\frac{dx}{\sqrt{x^2\pm a^2}}, \int\frac{dx}{a^2-x^2}, \int\frac{dx}{\sqrt{a^2-x^2}}, \int\frac{dx}{ax^2+bx+c}, \int\frac{dx}{\sqrt{ax^2+bx+c}}, \int\frac{(px+q)dx}{ax^2+bx+c}$

1. If $\vec{r}$ is equally inclined to the co-ordinate axis and magnitude of $\vec{r}$ is 6, then $\vec{r}$ equals

(a) $\sqrt{3}\,(1+\hat{i}+\hat{j})$ (b) $\frac{2(\hat{i}+\hat{j}+\hat{k})}{\sqrt{3}}$

(c) $\frac{(\hat{i}+\hat{j}+\hat{k})}{\sqrt{3}}$ (d) $\pm 2\sqrt{3}(\hat{i}+\hat{j}+\hat{k})$

2. The equation of the plane bisecting the acute angle between the planes $x - 2y + 2z + 3 = 0$ and $3x - 6y - 2z + 2 = 0$ is

(a) $2(8x-16y+4z)+27=0$

(b) $8x-16y+4z+27=0$

(c) $16x-32y+8z-27=0$

(d) $16x+32y+8z+27=0$

3. The distance between the planes $\vec{r}\cdot(2\hat{i}-3\hat{j}+6\hat{k})=5$ and $\vec{r}\cdot(6\hat{i}-9\hat{j}+18\hat{k})+20=0$ is

(a) 2/3 (b) 4/3 (c) 5/3 (d) 7/3

4. Let $\vec{n}$ be a vector of magnitude $2\sqrt{3}$ such that it makes acute equal angles with the co-ordinate axes, the vector form of equation of plane passing through $(1, -1, 2)$ and normal to $\vec{n}$ is

(a) $\vec{r}\cdot(\hat{i}+2\hat{j}+3\hat{k})=2$

(b) $\vec{r}\cdot(2\hat{i}-3\hat{j}+\hat{k})=2$

(c) $\vec{r}\cdot(2\hat{i}+\hat{j}+\hat{k})=2$

(d) $\vec{r}\cdot(\hat{i}+\hat{j}+\hat{k})=2$

By : Sankar Ghosh, S.G.M.C, Mob : 09831244397.

5. The S.D. between the lines
$\vec{r}=(5\hat{i}+7\hat{j}+3\hat{k})+\lambda(5\hat{i}-16\hat{j}+7\hat{k})$ and
$\vec{r}=(9\hat{i}+13\hat{j}+15\hat{k})+\lambda(3\hat{i}-8\hat{j}-5\hat{k})$ is
(a) 8 units (b) 5 units
(c) 10 units (d) none of these

6. The distance of the point (1, –2, 3) from the plane $x-y+z=5$ measured parallel to the line $\frac{x}{2}=\frac{y}{3}=\frac{z}{-6}$ is
(a) 5 (b) 3
(c) 1 (d) none of these

7. The distance from the point $-\hat{i}+2\hat{j}+6\hat{k}$ to the straight line passing through the point with position vector $2\hat{i}+3\hat{j}-4\hat{k}$ and parallel to the vectors $6\hat{i}+3\hat{j}-4\hat{k}$
(a) 10 (b) 7 (c) 5 (d) 3

8. The equation of the plane through the line of intersection of $\vec{r}\cdot(\hat{i}-2\hat{j}+2\hat{k})=1$ and $\vec{r}\cdot(2\hat{i}-\hat{j})+2=0$ and perpendicular to $\vec{r}\cdot(\hat{i}+\hat{j}+2\hat{k})+9=0$ is
(a) $\vec{r}\cdot(-5\hat{i}+\hat{j}+2\hat{k})=7$
(b) $\vec{r}\cdot(5\hat{i}+\hat{j}+2\hat{k})=7$
(c) $\vec{r}\cdot(5\hat{i}-\hat{j}-2\hat{k})=7$
(d) none of these

9. The reflection of the plane $2x-3y+4z-3=0$ in the plane $x - y + z - 3 = 0$ is the plane
(a) $4x - 3y + 2z - 15 = 0$
(b) $x - 3y + 2z - 15 = 0$
(c) $4x + 3y - 2z + 15 = 0$
(d) none of these

10. If $P(x, y, z)$ is a point on the line segment joining $A(2, 2, 4)$ and $B(3, 5, 6)$ such that projection of $\overrightarrow{OP}$ on axes are $\frac{13}{5},\frac{19}{5},\frac{26}{5}$ respectively, then P divides AB in the ratio
(a) 3 : 2 (b) 2 : 3 (c) 1 : 2 (d) 1 : 3

11. If $f(x) = x^\alpha \ln x$ and $f(0) = 0$ then the value of α for which Rolle's theorem can be applied in [0, 1] is
(a) –2 (b) –1 (c) 0 (d) 1/2

12. For all x in [0, 1], let the second derivative $f''(x)$ of a function exist and satisfy $|f''(x)| \le 1$. If $f(0) = f(1)$, then in (0, 1)
(a) $|f(x)| > 1$ (b) $|f(x)| < 1$
(c) $|f'(x)| > 1$ (d) $|f'(x)| < 1$

13. If $f(x)$ and $g(x)$ are differentiable functions in [0, 1] such that $f(0) = 2$, $f(1) = 6$, $g(0) = 0$, $g(1) = 2$, then there exists $0 < c < 1$ such that
(a) $f'(c) = g'(c)$ (b) $f'(c) = -g'(c)$
(c) $f'(c) = 2g'(c)$ (d) $2f'(c) = g'(c)$

14. If $f(x)=\int_{x^2}^{x^3}\frac{dt}{\log t}, x>0$, then
(a) $f(x)$ is maximum at $x = 1$
(b) $f(x)$ is an increasing function $\forall\ x \in R^+$ only
(c) $f(x)$ is minimum at $x = 1$
(d) $f(x)$ is neither maximum nor minimum at $x = 1$

15. If $y=f(x)=\int_1^x\sqrt{2-t^2}\,dt$. Then
(a) $f(x)$ increases if $|x|<\sqrt{2}$
(b) $f(x)$ decreases if $|x| < 2$
(c) $f(x)$ increases if $|x| > 2$
(d) none of these

16. A cone is inscribed in a sphere of radius r. The height of the cone when its volume is maximum is
(a) $2r$ (b) $\frac{\sqrt{3}r}{2}$ (c) $\frac{2r}{\sqrt{3}}$ (d) $\frac{4r}{3}$

17. Three normals are drawn from the point $(c, 0)$ to the curve $y^2 = x$. If two of the normals are perpendicular to each other, then $c =$
(a) 1/4 (b) 1/2
(c) 3/4 (d) 1

18. The curve $y=ax^3+bx^2+cx+5$ touches the x-axis at $P(-2, 0)$ and cut the y-axis at a point Q where its gradient is 3. Then, $a + b + c =$
(a) 9/4 (b) 7/4
(c) 5/4 (d) 3/4

19. The slope of the straight line which is both tangent and normal to the curve $4x^3 = 27y^2$ is
(a) ± 1 (b) $\pm\frac{1}{2}$ (c) $\pm\frac{1}{\sqrt{2}}$ (d) $\pm\sqrt{2}$

20. A spherical iron ball 10 cm in radius is coated with a layer of ice of uniform thickness that melts at a rate of 50 cm³/min. When the thickness of ice is 5 cm, then the rate at which the thickness of ice decreases, is
(a) $\frac{1}{18\pi}$ cm/min. (b) $\frac{1}{36\pi}$ cm/min.
(c) $\frac{5}{6\pi}$ cm/min. (d) $\frac{1}{54\pi}$ cm/min.

21. $\int \frac{dx}{\sin^2 x+\tan^2 x}=$

(a) $-\frac{1}{2}\cot x-\frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{\tan x}{\sqrt{2}}\right)+c$

(b) $\frac{1}{2}\cot x-\frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)+c$

(c) $\frac{1}{2}\cot x-\frac{1}{\sqrt{2}}\tan x+c$

(d) $-\cot x+\tan^{-1}\left(\frac{\tan x}{\sqrt{2}}\right)+c$

22. $\int \frac{dx}{(4+3x^2)\sqrt{3-4x^2}}=$

(a) $\frac{1}{5}\tan^{-1}\frac{2x}{\sqrt{3-4x^2}}+c$

(b) $\frac{1}{10}\tan^{-1}\left(\frac{5x}{2\sqrt{3-4x^2}}\right)+c$

(c) $\frac{1}{5}\tan^{-1}\frac{5x}{2\sqrt{3-4x^2}}+c$

(d) $\frac{1}{10}\tan^{-1}\left(\frac{5x}{\sqrt{3-4x^2}}\right)+c$

23. If $\int\sqrt{\tan x}\,dx=a\tan^{-1}\left(\frac{\tan x-1}{\sqrt{2\tan x}}\right)$

$+b\ln\left(\frac{\tan x+1+\sqrt{2\tan x}}{\tan x+1-\sqrt{2\tan x}}\right)+c$, then $a+b=$

(a) $\frac{1}{2}$ (b) $\frac{1}{\sqrt{2}}$ (c) $\frac{1}{2\sqrt{2}}$ (d) $\frac{1}{4}$

24. $\int \operatorname{cosec}\left(x-\frac{\pi}{6}\right)\operatorname{cosec}\left(x-\frac{\pi}{3}\right)dx=a\ln\left(\frac{\sin\left(x-\frac{\pi}{6}\right)}{\sin\left(x-\frac{\pi}{3}\right)}\right)+b,$

then a is equal to

(a) $\frac{2}{\sqrt{3}}$ (b) $-\frac{\sqrt{3}}{2}$ (c) 2 (d) -2

25. If $S_r=\int \sin x\,d(i^r x)$ where $(i=\sqrt{-1})$, then $\sum\limits_{r=1}^{4n-1} S_r$, is $(n\in N)$

(a) $\frac{-\cos x}{i^{4n}}+c$ (b) $\cos^2 x+c$

(c) 0 (d) none of these

26. The integral $\int\frac{dx}{\sqrt{x}+\sqrt[3]{x^2}}$ represents the function

(a) $6\left[\sqrt[3]{x^2}-\sqrt[3]{x}+\log\left|1+\sqrt[3]{x}\right|\right]+c$

(b) $3\sqrt[3]{x}-6\sqrt[6]{x}+6\log\left|1+\sqrt[6]{x}\right|+c$

(c) $3\sqrt[3]{x}+6\sqrt[6]{x}+6\log\left|1+\sqrt[6]{x}\right|+c$

(d) $6\sqrt[3]{x^2}-3\sqrt[3]{x}+6\log\left|1+\sqrt[3]{x}\right|+c$

27. $\int\frac{(x+1)}{x(1+xe^x)^2}dx=\log|1-f(x)|+f(x)+c$, then $f(x)$

(a) $\frac{1}{x+e^x}$ (b) $\frac{1}{1+xe^x}$

(c) $\frac{1}{(1+xe^x)^2}$ (d) $\frac{1}{(x+e^x)^2}$

28. $\int\frac{\cos^3 x+\cos^5 x}{\sin^2 x+\sin^4 x}dx$ is

(a) $\sin x-6\tan^{-1}(\sin x)+c$

(b) $\sin x-\frac{2}{\sin x}-6\tan^{-1}(\sin x)+c$

(c) $\sin x-\frac{2}{\sin x}+5\tan^{-1}(\sin x)+c$

(d) $\sin x+\frac{2}{\sin x}+6\tan^{-1}(\sin x)+c$

29. If $\int\sqrt{\frac{\cos x-\cos^3 x}{1-\cos^3 x}}dx=f(x)+c$, then $f(x)=$

(a) $\frac{3}{2}\sin^{-1}\left(\cos^{3/2}x\right)+k$

(b) $\frac{2}{3}\cos^{-1}\left(\cos^{3/2}x\right)+k$

(c) $\frac{2}{3}\sin^{-1}\left(\cos^{3/2}x\right)+k$

(d) none of these

30. If $\int f(x)\sin x\cos x\,dx=\frac{1}{2(b^2-a^2)}\log(f(x))+c$,

then $\int f(x)\,dx=$

(a) $\frac{1}{ab}\tan^{-1}\left(\frac{a\tan x}{b}\right)$ (b) $ab\tan^{-1}\left(\frac{a\tan x}{b}\right)$

(c) $\frac{1}{ab}\tan^{-1}\left(\frac{b\tan x}{a}\right)$ (d) $\frac{1}{ab}\tan^{-1}\left(\tan\left(\frac{bx}{a}\right)\right)$

SOLUTIONS

1. **(d) :** Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $\vec{r}$ makes equal angle θ to each of the axes.

$\therefore$ Direction cosines are $l = \cos\theta$, $m = \cos\theta$ and $n = \cos\theta$

Now, $\cos^2\theta + \cos^2\theta + \cos^2\theta = 1$

$\Rightarrow \quad 3\cos^2\theta = 1 \quad \Rightarrow \quad \cos\theta = \pm\dfrac{1}{\sqrt{3}}$

We know, $x = l|\vec{r}|$, $y = m|\vec{r}|$ and $z = n|\vec{r}|$

$\therefore \quad x = \pm\dfrac{1}{\sqrt{3}} \times 6 = \pm 2\sqrt{3} \qquad [\because |\vec{r}| = 6]$

$y = \pm\dfrac{1}{\sqrt{3}} \times 6 = \pm 2\sqrt{3}$ and $z = \pm 2\sqrt{3}$

So, $r = x\hat{i} + y\hat{j} + z\hat{k} = \pm 2\sqrt{3}(\hat{i} + \hat{j} + \hat{k})$

2. **(a) :** The given equation of the planes are

$x - 2y + 2z + 3 = 0 \qquad \therefore \quad (a_1, b_1, c_1) = (1, -2, 2)$

and $3x - 6y - 2z + 2 = 0 \;\therefore\; (a_2, b_2, c_2) = (3, -6, -2)$

Now, $a_1a_2 + b_1b_2 + c_1c_2 = 3 + 12 - 4 = 11 > 0$

$\therefore$ Equation of acute angle bisector is

$$\frac{x - 2y + 2z + 3}{3} = -\frac{(3x - 6y - 2z + 2)}{7}$$

$\Rightarrow \quad 16x - 32y + 8z + 27 = 0$

3. **(c) :** The given equation of planes are

$\vec{r}\cdot(2\hat{i} - 3\hat{j} + 6\hat{k}) = 5$ and $\vec{r}\cdot(6\hat{i} - 9\hat{j} + 18\hat{k}) + 20 = 0$

Now changing to cartesian form of the above plane we get, $2x - 3y + 6z - 5 = 0$ and $6x - 9y + 18z + 20 = 0$ respectively.

Now distance of the plane $6x - 9y + 18z + 20 = 0$ from a point (x_1, y_1, z_1) which lies on the plane $2x - 3y + 6z - 5 = 0$ is

$$\frac{|6x_1 - 9y_1 + 18z_1 + 20|}{\sqrt{6^2 + (-9)^2 + 18^2}}$$

$$= \frac{|3(2x_1 - 3y_1 + 6z_1 - 5) + 15 + 20|}{21} = \frac{0 + 15 + 20}{21}$$

$$= \frac{35}{21} = \frac{5}{3} \text{ units}$$

4. **(d) :** Given that $\alpha = \beta = \gamma \;\Rightarrow\; \cos\alpha = \cos\beta = \cos\gamma$

i.e. $\quad l = m = n \quad \therefore \quad l = \dfrac{1}{\sqrt{3}} \quad (\because l^2 + m^2 + n^2 = 1)$

$\therefore \quad \vec{n} = 2\sqrt{3}\left(\dfrac{\hat{i} + \hat{j} + \hat{k}}{\sqrt{3}}\right) \qquad [\because \vec{r} = |\vec{r}|(l\hat{i} + m\hat{j} + n\hat{k})]$

Thus the required vector equation of plane is

$$\vec{r}\cdot(2\hat{i} + 2\hat{j} + 2\hat{k}) = (\hat{i} - \hat{j} + 2\hat{k})(2\hat{i} + 2\hat{j} + 2\hat{k})$$

$\Rightarrow \quad \vec{r}\cdot(\hat{i} + \hat{j} + \hat{k}) = 2$

5. **(d) :** The given lines are

$$\vec{r} = (5\hat{i} + 7\hat{j} + 3\hat{k}) + \lambda(5\hat{i} - 16\hat{j} + 7\hat{k})$$

and $\vec{r} = (9\hat{i} + 13\hat{j} + 15\hat{k}) + \lambda(3\hat{i} - 8\hat{j} - 5\hat{k})$

Here $\vec{a}_1 = 5\hat{i} + 7\hat{j} + 3\hat{k}$, $\vec{a}_2 = 9\hat{i} + 13\hat{j} + 15\hat{k}$

and $\vec{b}_1 = 5\hat{i} - 16\hat{j} + 7\hat{k}$, $\vec{b}_2 = 3\hat{i} - 8\hat{j} - 5\hat{k}$

Now, $\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5 & -16 & 7 \\ 3 & -8 & -5 \end{vmatrix} = 136\hat{i} + 46\hat{j} + 8\hat{k}$

and $(\vec{a}_2 - \vec{a}_1) = 4\hat{i} + 6\hat{j} + 12\hat{k}$

$\therefore \quad \text{S.D.} = \left|\dfrac{(\vec{b}_1 \times \vec{b}_2)\cdot(\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|}\right| = \dfrac{916}{143.8} = 6.37$

6. **(c) :** The equation of the line passing through the point $(1, -2, 3)$ and parallel to the line $\dfrac{x}{2} = \dfrac{y}{3} = \dfrac{z}{-6}$ is

$\dfrac{x - 1}{2} = \dfrac{y + 2}{3} = \dfrac{z - 3}{-6} = \lambda$ (say) $\qquad$ ...(i)

$\therefore \quad x = 2\lambda + 1$, $y = 3\lambda - 2$ and $z = -6\lambda + 3$

Now point of intersection of the line (i) and plane $x - y + z = 5$ $\qquad$ ...(ii)

can be obtained by putting $(2\lambda + 1, 3\lambda - 2, -6\lambda + 3)$ in (ii), which gives

$$(2\lambda + 1) - (3\lambda - 2) + (-6\lambda + 3) = 5$$

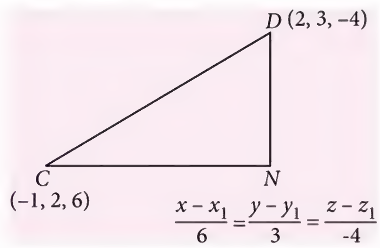
$\Rightarrow \quad -7\lambda = -1 \quad \Rightarrow \quad \lambda = \dfrac{1}{7}$

and hence the required point of intersection of line and plane is

$$(x, y, z) \equiv \left(\frac{9}{7}, \frac{-11}{7}, \frac{15}{7}\right)$$

7. **(b) :** Let C is a point with position vector $-\hat{i} + 2\hat{j} + 6\hat{k}$ and D with position vector $2\hat{i} + 3\hat{j} - 4\hat{k}$

$\therefore \quad CD = 3\hat{i} + \hat{j} - 10\hat{k}$



CN = Projection of CD on CN
$= l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)$

where l, m and n are $\frac{6}{\sqrt{61}}, \frac{3}{\sqrt{61}}, \frac{-4}{\sqrt{61}}$

$$= 6\times\frac{3}{\sqrt{61}} + \frac{3}{\sqrt{61}}\times 1 + \frac{(-10)(-4)}{\sqrt{61}} = \sqrt{61}$$

$$\therefore \quad DN = \sqrt{(CD)^2 - (CN)^2} = \sqrt{3^2 + 1^2 + (-10)^2 - 61} = 7$$

8. (a) : The equation of plane through the line of intersection of $\vec{r}\cdot(\hat{i} - 2\hat{j} + 2\hat{k}) = 1$ and $\vec{r}\cdot(2\hat{i} - \hat{j}) + 2 = 0$ is

$$[\vec{r}\cdot(\hat{i} - 2\hat{j} + 2\hat{k}) - 1] + \lambda[\vec{r}\cdot(2\hat{i} - \hat{j}) + 2] = 0$$

$$\Rightarrow \quad \vec{r}\cdot[(1+2\lambda)\hat{i} + (-2-\lambda)\hat{j} + 2\hat{k}] = 1 - 2\lambda \qquad \text{...(i)}$$

Since (i) is perpendicular to $\vec{r}\cdot(\hat{i} + \hat{j} + 2\hat{k}) + 9 = 0$

$\therefore \quad 1(1 + 2\lambda) + (-2 - \lambda)1 + 2 \cdot 2 = 0 \quad \Rightarrow \quad \lambda = -3$

Required equation of plane is $\vec{r}\cdot(-5\hat{i} + \hat{j} + 2\hat{k}) = 7$

9. (a) : As we know that reflection of $a'x + b'y + c'z + d' = 0$ in the plane $ax + by + cz + d = 0$ is given by $2(aa' + bb' + cc')(ax + by + cz + d)$
$= (a^2 + b^2 + c^2)(a'x + b'y + c'z + d')$

$\therefore \quad 2(2 + 3 + 4)(x - y + z - 3) = 3(2x - 3y + 4z - 3)$

$\Rightarrow \quad 6(x - y + z - 3) = 2x - 3y + 4z - 3$

$\Rightarrow \quad 4x - 3y + 2z - 15 = 0$ is the required equation of the plane.

10. (a) : Projection of $\overrightarrow{OP}$ on coordinate axis are $\frac{13}{5}, \frac{19}{5}, \frac{26}{5}$

So $\overrightarrow{OP} = \frac{13}{5}\hat{i} + \frac{19}{5}\hat{j} + \frac{26}{5}\hat{k}$

Let P divides AB in the ratio $\lambda : 1$

$$\therefore \quad P\left[\frac{3\lambda+2}{\lambda+1}, \frac{5\lambda+2}{\lambda+1}, \frac{6\lambda+4}{\lambda+1}\right]$$

$$\therefore \quad \frac{3\lambda+2}{\lambda+1} = \frac{13}{5} \Rightarrow 2\lambda = 3 \Rightarrow \lambda = \frac{3}{2}$$

$\therefore \quad$ Required ratio is $\lambda : 1$ *i.e.* $3 : 2$

11. (d) : Here $f(0) = f(1) = 0$ and $f(x)$ is differentiable in $[0, 1]$.

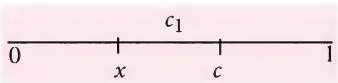
$f'(x) = x^{\alpha - 1}(1 + \alpha \ln x)$

$f(x)$ is continuous in $[0, 1]$

Now, $f'(x) = 0$, only if $\alpha > 0$, $x \to 0^+$.

12. (d) : $f(0) = f(1)$

$\Rightarrow \quad f'(c) = 0, \; 0 < c < 1$



By Rolle's theorem

Let x be any point in $[0, 1]$ other then c

By Lagranges theorem

$f'(x) - f'(c) = (x - c)\,|f''(c_1)|, \; x < c$

$|f'(x)| = |x - c|\,|f''(c_1)| < 1$

Since $|x - c| < 1$, $|f''(c_1)| \le 1$

13. (c) : Let $F(x) = f(x) - 2g(x)$

$F(0) = f(0) - 2g(0) = 2$

$F(1) = f(1) - 2g(1) = 2$

$F(x)$ satisfies the condition for Rolle's theorem. So, there exist c, such that $F'(c) = 0$

or $\quad f'(c) - 2g'(c) = 0$ for $0 < c < 1$.

14. (d) : Here $f(x) = \int_{x^2}^{x^3} \frac{1}{\log t}\,dt$

$$\therefore \quad f'(x) = \frac{3x^2}{3\log x} - \frac{2x}{2\log x}$$

$$\Rightarrow \quad f'(x) = \frac{x^2 - x}{\log x} = \frac{x(x-1)}{\log x}$$

For extremum, we put $f'(x) = 0$.

Now, $f'(x) = 0$ gives $x(x - 1) = 0$

$\therefore \quad$ Critical points are $x = 0$ and $x = 1$

Let us consider $x = 1$ as $x > 0$

Now for $x < 1$, $f'(x) = (-\text{ve})(-\text{ve}) = (+)\text{ve} > 0$

and for $x > 1$, $f'(x) = (+\text{ve})(+\text{ve}) = (+)\text{ve} > 0$

$\therefore \quad f'(x)$ does not change its sign in the immediate neighbourhood of $x = 1$. So $x = 1$ is neither the point of maxima nor minima.

15. (a) : $f(x) = \int_1^x \sqrt{2 - t^2}\,dt$

$$f'(x) = \sqrt{2 - x^2}\,\frac{d}{dx}(x) = \sqrt{2 - x^2}$$

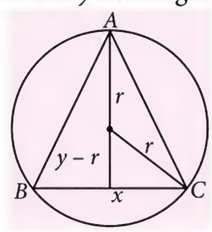
For an increasing function

$$f'(x) > 0 \quad \Rightarrow \quad \sqrt{2 - x^2} > 0 \quad \Rightarrow \quad 2 - x^2 > 0$$

$$\Rightarrow \quad (\sqrt{2} + x)(\sqrt{2} - x) > 0$$

$$\Rightarrow \quad (x - \sqrt{2})(x + \sqrt{2}) < 0 \quad \Rightarrow |x| < \sqrt{2}$$

16. (d) : Let x be the radius of the base and y the height of the cone.



$x^2 = r^2 - (y - r)^2 = 2ry - y^2$

$$V = \frac{4}{3}\pi x^2 y = \frac{4}{3}\pi(2ry^2 - y^3)$$

$$\frac{dV}{dy} = 0 \Rightarrow 4ry = 3y^2 \Rightarrow y = \frac{4r}{3}$$

17. (c) : $x = t^2,\ y = t,\ \dfrac{dy}{dx} = \dfrac{1}{2t}$

Equation of normal is

$$\frac{1}{2t}(y - t) + x - t^2 = 0$$

It passes through $(c, 0) \Rightarrow t^2 + \dfrac{1}{2} - c = 0$

t_1, t_2 are roots $\Rightarrow t_1 t_2 = \dfrac{1}{2} - c,$

Slopes $m_1 = \dfrac{1}{2t_1},\ m_2 = \dfrac{1}{2t_2}$

$$m_1 m_2 = -1 \Rightarrow \frac{1}{4t_1 t_2} = -1 = 2 - 4c \Rightarrow c = \frac{3}{4}$$

18. (b) : $y = f(x)\ \therefore\ f(-2) = f'(-2) = 0,\ f'(0) = 3$

$f(0) = 5$

$$f(x) = (x+2)^2\left(ax + \frac{5}{4}\right)$$

$$f'(x) = 2(x+2)\left(ax + \frac{5}{4}\right) + (x+2)^2 a$$

$$f'(0) = 3 \Rightarrow 5 + 4a = 3 \Rightarrow a = -\frac{1}{2}$$

$$f(x) = (x+2)^2\left(-\frac{x}{2} + \frac{5}{4}\right)$$

$$\therefore\ a + b + c + 5 = f(1) = \frac{27}{4} \Rightarrow a + b + c = \frac{7}{4}$$

19. (d) : The given equation of the curve is $4x^3 = 27y^2$

$\Rightarrow x = 3t^2;\ y = 2t^3$

$\therefore\ \dfrac{dy}{dx} = t$

The tangent at t, $y - 2t^3 = t(x - 3t^2)$

$\Rightarrow\ tx - y = t^3$ ....(i)

The normal at t_1,

$t_1 y + x = 2t_1^4 + 3t_1^2$ ...(ii)

As (i) and (ii) are identical

$$\therefore\ \frac{t}{1} = -\frac{1}{t_1} = \frac{t^3}{2t_1^4 + 3t_1^2}$$

$\Rightarrow\ t_1 = -\dfrac{1}{t}$ and $-t^3 = 2t_1^3 + 3t_1$

Eliminating t_1, we get $t^6 = 2 + 3t^2$

$\Rightarrow\ t^2 = 2,\ t = \pm\sqrt{2}$

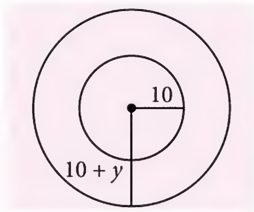
The lines are $y = \pm\sqrt{2}(x - 2)$.

20. (a) : $v = \dfrac{4}{3}\pi(y + 10)^3$ where y is thickness of ice

$$\therefore\ \frac{dv}{dt} = 4\pi(10 + y)^2\frac{dy}{dt} \Rightarrow \left(\frac{dy}{dt}\right)_{t=5} \frac{50}{4\pi(15)^2}$$

$$\left[\because\ \frac{dv}{dt} = 50\ \text{cm}^3/\text{min}\right]$$

$$= \frac{1}{18\pi}\ \text{cm/min.}$$



21. (a) : Let $I = \displaystyle\int \frac{dx}{\sin^2 x + \tan^2 x} = \int \frac{\sec^2 x dx}{\tan^2 x(2 + \tan^2 x)}$

Put $\tan x = t$

$$\int \frac{dt}{t^2(2 + t^2)} = \frac{1}{2}\int\left(\frac{1}{t^2} - \frac{1}{2 + t^2}\right)dt$$

$$= -\frac{1}{2t} - \frac{1}{2\sqrt{2}}\tan^{-1}\frac{t}{\sqrt{2}} + c$$

$$= -\frac{1}{2}\cot x - \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{1}{\sqrt{2}}\tan x\right) + c$$

22. (b) : Let $I = \displaystyle\int \frac{dx}{(4 + 3x^2)\sqrt{3 - 4x^2}}$

$$= 2\int \frac{d\theta}{9\sin^2\theta + 16} \qquad \left[\text{Putting } x = \frac{\sqrt{3}}{2}\sin\theta\right]$$

$$= 2\int \frac{\sec^2\theta d\theta}{9\tan^2\theta + 16\sec^2\theta} = 2\int \frac{\sec^2\theta d\theta}{16 + 25\tan^2\theta}$$

$$= 2\int \frac{d(\tan\theta)}{(4)^2 + (5\tan\theta)^2} = 2 \times \frac{1}{5} \times \frac{1}{4}\tan^{-1}\left(\frac{5}{4}\tan\theta\right) + c$$

$$= \frac{1}{10}\tan^{-1}\left(\frac{5}{4}\tan\theta\right) + c = \frac{1}{10}\tan^{-1}\left(\frac{5x}{2\sqrt{3 - 4x^2}}\right) + c$$

23. (c) : Let $I = \displaystyle\int \sqrt{\tan x}\, dx$

Put $\tan x = t^2$

$$I = 2\int \frac{t^2 dt}{t^4 + 1} = \int \frac{t^2 + 1 + t^2 - 1}{t^4 + 1}dt$$

$$= \int \frac{1 + \frac{1}{t^2}}{t^2 + \frac{1}{t^2}}dt + \int \frac{1 - \frac{1}{t^2}}{t^2 + \frac{1}{t^2}}dt = \int \frac{d\left(t - \frac{1}{t}\right)}{\left(t - \frac{1}{t}\right)^2 + 2} + \int \frac{d\left(t + \frac{1}{t}\right)}{\left(t + \frac{1}{t}\right)^2 - 2}$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t - \frac{1}{t}}{\sqrt{2}}\right) + \frac{1}{2\sqrt{2}}\ln\left(\frac{t + \frac{1}{t} - \sqrt{2}}{t + \frac{1}{t} + \sqrt{2}}\right) + c$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan x - 1}{\sqrt{2\tan x}}\right) - \frac{1}{2\sqrt{2}}\ln\left(\frac{\tan x + 1 + \sqrt{2\tan x}}{\tan x + 1 - \sqrt{2\tan x}}\right) + c$$

$$\therefore\ a + b = \frac{1}{\sqrt{2}} - \frac{1}{2\sqrt{2}} = \frac{1}{2\sqrt{2}}$$

24. (c) : Let $I=\int \operatorname{cosec}\left(x-\frac{\pi}{6}\right)\operatorname{cosec}\left(x-\frac{\pi}{3}\right)dx$

$$=\int \frac{2\sin\frac{\pi}{6}}{\sin\left(x-\frac{\pi}{6}\right)\sin\left(x-\frac{\pi}{3}\right)}dx$$

$$=2\int \frac{\sin\left\{\left(x-\frac{\pi}{6}\right)-\left(x-\frac{\pi}{3}\right)\right\}}{\sin\left(x-\frac{\pi}{6}\right)\sin\left(x-\frac{\pi}{3}\right)}dx$$

$$=2\int\left[\cot\left(x-\frac{\pi}{3}\right)-\cot\left(x-\frac{\pi}{6}\right)\right]dx$$

$$=2\ln\left\{\frac{\sin\left(x-\frac{\pi}{3}\right)}{\sin\left(x-\frac{\pi}{6}\right)}\right\}+b$$

25. (d)

26. (b) : Let $I=\int \frac{dx}{\sqrt{x}+3\sqrt{x^2}}$ [put $x=t^6$]

$$=\int \frac{6t^5dt}{t^3+t^4}=6\int\frac{t^2dt}{t+1}=6\int\left(t-1+\frac{1}{t+1}\right)dt$$

$= 3t^2 - 6t + 6\log|t + 1| + c$

$=3\sqrt[3]{x}-6\sqrt[6]{x}+6\log|\sqrt[6]{x}+1|+c$

27. (b) : Let $I=\int\frac{(x+1)}{x(1+xe^x)^2}dx$

Put $xe^x=t \Rightarrow e^x(1+x)dx=dt$

$\therefore\quad I=\int\frac{dt}{t(1+t)^2}$

Now, let $\frac{1}{t(1+t)^2}=\frac{A}{t}+\frac{B}{1+t}+\frac{C}{(1+t)^2}$

$\Rightarrow\quad 1 = A(1+t)^2 + Bt(1+t) + Ct$

Put $t = 0$, then, $1 = A$

Put $t = -1$, then $C = -1$

Equating the coefficients of t^2, we get

$0 = A + B \Rightarrow B = -A = -1$

$\therefore\quad I=\int\frac{1}{t}dt-\int\frac{dt}{1+t}-\int\frac{dt}{(1+t)^2}$

$$=\log|t|-\log|1+t|+\frac{1}{1+t}+c$$

$$=\log\left|\frac{t}{1+t}\right|+\frac{1}{1+t}+c=\log\left|\left(1-\frac{1}{1+t}\right)\right|+\frac{1}{1+t}+c$$

28. (b) : Let $I=\int\frac{\cos^3x+\cos^5x}{\sin^2x+\sin^4x}dx$

$$=\int\frac{\cos x\{(1-\sin^2x)+(1-\sin^2x)^2\}}{\sin^2x+\sin^4x}dx$$

Put $\sin x = t$

$$I=\int\frac{(1-t^2)+(1-t^2)^2}{t^2+t^4}dt=\int\frac{2+t^4-3t^2}{t^2+t^4}dt$$

$$=\int\frac{(t^2-1)(t^2-2)}{t^2+t^4}dt$$

$$=\int\left(1+\frac{2}{t^2}-\frac{6}{1+t^2}\right)dt\ \text{(using Partial fractions)}$$

$$=t-\frac{2}{t}-6\tan^{-1}t+c=\sin x-\frac{2}{\sin x}-6\tan^{-1}(\sin x)+c$$

29. (b) : Let $I=\int\frac{\sqrt{\cos x(1-\cos^2x)}}{\sqrt{1-\cos^3x}}dx$

$$=\int\frac{\cos^{1/2}x\sin x}{\sqrt{1-(\cos^{3/2}x)^2}}dx$$

Put $t=\cos^{3/2}x$

$$I=-\frac{2}{3}\int\frac{1}{\sqrt{1-t^2}}dt=-\frac{2}{3}\sin^{-1}(\cos^{3/2}x)+c$$

$$=-\frac{2}{3}\left(\frac{\pi}{2}-\cos^{-1}\cos^{3/2}x\right)+c=\frac{2}{3}\cos^{-1}(\cos^{3/2}x)+k$$

30. (a) : Let $I=\int f(x)\sin x\cos x\,dx$

$$=\frac{1}{2(b^2-a^2)}\log f(x)+c$$

$$\therefore\ f(x)\sin x\cos x=\frac{1}{2(b^2-a^2)}\left[\frac{f'(x)}{f(x)}\right]+c$$

$$\Rightarrow\ 2(b^2-a^2)\sin x\cos x=\frac{f'(x)}{((f(x))^2}$$

$$\Rightarrow\ 2\int(b^2-a^2)\sin x\cos x\,dx=\frac{-1}{f(x)}$$

$$\Rightarrow\ b^2\int 2\sin x\cos x dx-a^2\int 2\sin x\cos x\,dx=-\frac{1}{f(x)}$$

$$\Rightarrow\ -b^2\cos^2x-a^2\sin^2x=-\frac{1}{f(x)}$$

$$\Rightarrow\ f(x)=\frac{1}{a^2\sin^2x+b^2\cos^2x}=\frac{\sec^2x}{b^2+a^2\tan^2x}$$

$$\therefore\ \int f(x)dx=\int\frac{\sec^2x}{b^2+a^2\tan^2x}dx=\frac{1}{ab}\tan^{-1}\left(\frac{a\tan x}{b}\right)$$

JEE WORK OUTS

TWO DIMENSIONAL GEOMETRY

PAPER-I

ONE OF MORE THAN ONE OPTION(S) CORRECT TYPE

This section contains 10 multiple choice questions. Each question has four choices (a), (b), (c) and (d) out of which ONE or MORE may be correct. [Correct ans. 3 marks & wrong ans., no negative mark]

1. A curve that passing through (2, 4) and having subnormal of constant length of 8 units can be
   (a) $y^2 = 16x - 16$ (b) $y^2 = -16x + 48$
   (c) $x^2 = 16y - 60$ (d) $x^2 = -16y + 68$

2. If the conjugate of $(x + iy)(1 - 2i)$ be $1 + i$, then
   (a) $x=\frac{1}{5}$ (b) $x+iy=\frac{1}{5}(3+i)$
   (c) $x-iy=\frac{1}{5}(3+i)$ (d) $x+iy=\frac{1-i}{1-2i}$

3. If $a + b + c = 0$, then the roots of the equation $(b + c - a)x^2 + (c + a - b)x + (a + b - c) = 0$ can be
   (a) imaginary (b) real and equal
   (c) real and unequal (d) none of these

4. The solution of $\left(\frac{dy}{dx}\right)^2+2y\cot x\frac{dy}{dx}=y^2$ is
   (a) $y-\frac{c}{1+\cos x}=0$ (b) $y=\frac{c}{1-\cos x}$
   (c) $x=2\sin^{-1}\left(\sqrt{\frac{c}{2y}}\right)$ (d) none of these

5. $\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{j} 1$ is equal to
   (a) $\frac{n(n+1)(n+2)}{6}$ (b) Σn^2
   (c) ${}^nC_3$ (d) ${}^{n+2}C_3$

6. If A and B are two matrices such that $AB = BA$, then $\forall\, n \in N$
   (a) $A^nB = BA^n$ (b) $(AB)^n = A^nB^n$
   (c) $(A + B)^n = {}^nC_0A^n + {}^nC_1A^{n-1}B + {}^nC_2A^{n-2}B^2 + \ldots + {}^nC_n B^n$
   (d) $A^{2n} - B^{2n} = (A^n - B^n)(A^n + B^n)$

7. If the graph of the function $f(x)$ is symmetrical about two lines $x = a$ and $x = b$ then $f(x)$ must be periodic with period
   (a) $\frac{b-a}{2}$ (b) $b - a$
   (c) $2(b - a)$ (d) none of these

8. The function $f(x)=\begin{cases}|x-3| , & x\geq 1\\ \frac{x^2}{4}-\frac{3x}{2}+\frac{13}{4}, & x<1\end{cases}$ is
   (a) continuous at $x = 1$
   (b) differentiable at $x = 1$
   (c) continuous at $x = 3$
   (d) differentiable at $x = 3$

9. Let $f(x) = \sin(\pi x) - 4x(1 - x)$, then
   (a) $\frac{\sin(\pi x)}{x(1-x)}>4\ \forall\, x\in(0,1)$
   (b) $f'\left(\frac{5}{8}\right)+f'\left(\frac{3}{8}\right)=0$

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(c) $f''(x) = 0$ has no solution in (0, 1)

(d) Rolle's theorem cannot be applied to $f'(x)$ in $\left[\alpha, \frac{1}{2}\right]$ for some $\alpha \in \left(0, \frac{1}{2}\right)$

10. Let $\int \sqrt{\frac{x}{1-x^3}}\, dx = \frac{2}{3} g(f(x)) + c$. Then

(a) $g(x) = \cot^{-1}(x)$ and $f(x) = \sqrt{\frac{1-x^3}{x^3}}$

(b) $g(x) = \tan^{-1}(x)$ and $f(x) = \sqrt{\frac{1-x^3}{x^3}}$

(c) $g(x) = \cot^{-1}(x)$ and $f(x) = \sqrt{\frac{x^3}{1-x^3}}$

(d) $g(x) = \tan^{-1}(x)$ and $f(x) = \sqrt{\frac{x^3}{1-x^3}}$

INTEGER ANSWER TYPE

This section contains 10 questions. Each question, when worked out will result in one integer from 0 to 9 (both inclusive). [Correct ans. 3 marks & wrong ans., no negative mark]

11. If α, β, γ are the roots of the equation $x^3 - 9x^2 + 14x + 24 = 0$, then find the value of $|\alpha + \beta + \gamma + \alpha\beta + \beta\gamma + \gamma\alpha + \alpha\beta\gamma|$.

12. The least degree of a polynomial with integer coefficient whose one of the roots may be $\cos 12°$ is

13. If $\sin^{-1}x + \sin^{-1}y = \pi$ and, if $x = \lambda y$, then the value of $39^{2\lambda} + 5^{\lambda} - 1525$ must be

14. The number of solutions of the equation

$$\int_{-1}^{x}\left(8t^2 + \frac{28t}{3} + 4\right)dt = \frac{\frac{3}{2}x+1}{\log_{x+1}\sqrt{x+1}}$$ is

15. In a ΔABC, if a is the arithmetic mean and b, c are two geometric means between any two positive numbers. Then $\frac{\sin^3 B + \sin^3 C}{\sin A \sin B \sin C}$ is equal to

16. Let m denotes the number of ways in which 4 different balls of green colour and 4 different balls of red colour can be distributed equally among 4 persons if each person has balls of the same colour and n be corresponding figure when all the four persons have balls of different colour. Find $\frac{(m+n)}{132}$.

17. Let $f(x) = [3 + 4 \sin x]$ (where [ ] denotes the greatest integer function). If sum of all the values of x in $[\pi, 2\pi]$ where $f(x)$ fails to be differentiable, is $\frac{k\pi}{2}$, then the value of $\frac{k}{8}$ is

18. The polynomial $p(x) = 1 - x + x^2 - x^3 + ... + x^{16} - x^{17}$ can be written as a polynomial in y where $y = x + 1$, then let coefficient of y^2 be k then $\left[\frac{k}{400}\right]$ (where [.] denote greatest integer function) is _____

19. Let A be the centre of the circle $x^2 + y^2 - 2x - 4y - 20 = 0$. The tangents at the points $B(1, 7)$ and $C(4, -2)$ on the circle meet at the point D. If Δ denotes the area of the quadrilateral $ABDC$, then $\sqrt{\frac{\Delta}{3}}$ is equal to

20. Find the integer n for which $\lim_{x\to 0} \frac{(\cos x - 1)(\cos x - e^x)}{x^n}$ is a finite nonzero number.

PAPER-II

ONLY ONE OPTION CORRECT TYPE

This section contains 10 multiple choice questions. Each question has four choices (a), (b), (c) and (d) for its answer, out of which ONLY ONE is correct. [Correct ans. 3 marks and wrong ans. –1]

1. If x and y are positive real number and m, n are any positive integers and $E = \frac{x^n y^m}{(1+x^{2n})(1+y^{2m})}$ then

(a) $E > \frac{1}{4}$ (b) $E > \frac{1}{2}$ (c) $E \le \frac{1}{4}$ (d) $E < \frac{1}{8}$

2. Four persons are selected at random out of 3 men, 2 women and 4 children. What is the chance that exactly 2 of them are children?

(a) $\frac{9}{21}$ (b) $\frac{10}{23}$ (c) $\frac{11}{24}$ (d) $\frac{10}{21}$

3. For the equation $x^2 - (k + 1)x + (k^2 + k - 8) = 0$ if one root is greater than 2 and other is less than 2, then k can take any value between

(a) $(-2, 3)$ (b) $\left(-\frac{11}{3}, 3\right)$

(c) $\left(-\frac{11}{3},-2\right)$ (d) (0, 5)

4. If z is a complex number satisfying $|z+1-i| \le 1$, then the maximum value of $|z|$ is

(a) $\sqrt{2}$ (b) $\sqrt{2}-1$

(c) $\sqrt{2}+1$ (d) 1

5. The value of $\int_1^{e^{37}} \frac{\pi \sin(\pi \ln x)}{x} dx$ is

(a) 1 (b) −1 (c) 2 (d) 4

6. For a parabola having focus at S, vertex at A such that $SA = l_1$ units and focal chord PQ of length l_2 units is given, then $ar(\Delta APQ)$ is

(a) $l_1 l_2$ (b) $l_1\sqrt{l_1 l_2}$ (c) $l_2\sqrt{l_1 l_2}$ (d) $\frac{l_2^3}{2l_1}$

7. If $\cos^4\theta \sec^2\alpha, \frac{1}{2}$ and $\sin^4\theta \operatorname{cosec}^2\alpha$ are in A.P., then $\cos^8\theta \sec^6\alpha, \frac{1}{2}$ and $\sin^8\theta \operatorname{cosec}^6\alpha$ are in

(a) A.P. (b) G.P. (c) H.P. (d) A.G.P.

8. The point of intersection of the plane $\vec{r}\cdot 3\hat{i}-5\hat{j}+2\hat{k}=6$ with the straight line passing through the origin and perpendicular to the plane $2x - y - z = 4$ is

(a) (1, −1, −1) (b) (−1, −1, 2)

(c) (4, 2, 2) (d) $\left(\frac{4}{3},\frac{-2}{3},\frac{-2}{3}\right)$

9. The function $f(x)=\log_e(1+x)-\frac{2x}{2+x}$ is increasing on

(a) $(-1, \infty)$ (b) $(-\infty, 0)$

(c) $(-\infty, \infty)$ (d) none of these

10. If $\lim_{x\to\infty} f(x^2)=a$ (a finite number), then which of the following is/are true?

(a) $\lim_{x\to\infty} x^2 f'(x^2)=0$ (b) $\lim_{x\to\infty} x^2 f'(x^2)=2a$

(c) $\lim_{x\to\infty} x^4 f''(x^2)=0$ (d) Both (a) and (c)

COMPREHENSION TYPE

This section contains 3 paragraph. Based upon each of the paragraphs 2 multiple choice questions have to be answered Each of these questions has four choices (a), (b), (c) and (d) out of which ONLY ONE is correct. [Correct ans. 3 marks & wrong ans. −1]

Paragraph for Q. No. 11 and 12

Suppose equation is $f(x) - g(x) = 0$ or $f(x) = g(x) = y$ say, then draw the graphs of $y = f(x)$ and $y = g(x)$. If graphs of $y = f(x)$ and $y = g(x)$ cuts at one, two, three, ..., no points, then number of solutions are one, two, three, ..., zero respectively.

11. The number of solutions of $\sin x = \frac{|x|}{10}$ is

(a) 4 (b) 6 (c) 8 (d) 10

12. Total number of solutions of the equation $3x + 2\tan x = \frac{5\pi}{2}$ in $x \in [0, 2\pi]$ is equal to

(a) 1 (b) 2 (c) 3 (d) 4

Paragraph for Q. No. 13 and 14

A and B are two matrices of same order 3 × 3, where

$$A=\begin{pmatrix}1&2&3\\2&3&4\\5&6&8\end{pmatrix} \text{and } B=\begin{pmatrix}3&2&5\\2&3&8\\7&2&9\end{pmatrix}$$

13. The value of $adj\,(adj\,A)$ is equal to

(a) $2A$

(b) $4A$

(c) $8A$

(d) none of these

14. The value of $\frac{|adj(adj\,B)|}{(24)^4}$ is equal to

(a) 9 (b) 16

(c) 25 (d) 1

Paragraph for Q. No. 15 and 16

Consider two functions $f(x)=\lim_{n\to\infty}\left(\cos\frac{x}{\sqrt{n}}\right)^n$ and $g(x) = -x^{4b}$, where $b=\lim_{x\to\infty}\left(\sqrt{x^2+x+1}-\sqrt{x^2+1}\right)$.

15. The function $f(x)$ is

(a) e^{-x^2} (b) $e^{-\frac{x^2}{2}}$ (c) e^{x^2} (d) $e^{\frac{x^2}{2}}$

16. Number of solutions of $f(x) + g(x) = 0$ is

(a) 0 (b) 1 (c) 2 (d) 4

MATRIX MATCH TYPE

This section contains 4 questions, each having two matching columns. Choices for the correct combination of elements from column-I and column-II are given as options (a), (b), (c) and (d), out of which one is correct. [Correct ans. 3 marks & wrong ans. −1]

17. Match the following.

	Column-I	Column-II	
(P)	The value of $\sin(\sin^{-1}1)$ is	1.	-1
(Q)	The points $(k, 2-2k)$, $(-k+1, 2k)$ and $(-4-k, 6-2k)$ are collinear if k is equal to	2.	1
(R)	The value of $\frac{1}{3}\left[\cos^4\left(\frac{\pi}{8}\right)+\cos^4\left(\frac{3\pi}{8}\right)+\cos^4\left(\frac{5\pi}{8}\right)+\cos^4\left(\frac{7\pi}{8}\right)\right]$ is	3.	1/2
(S)	If a, b, c are all different from zero, and $\Delta=\begin{vmatrix}1+a & 1 & 1\\ 1 & 1+b & 1\\ 1 & 1 & 1+c\end{vmatrix}=0$ then the value of $a^{-1}+b^{-1}+c^{-1}$ is	4.	$\sqrt{2}\pm1$

Codes :

	P	Q	R	S		P	Q	R	S
(a)	4	1, 4	2	3	(b)	2	1	2, 4	3
(c)	1, 3	3	2	4	(d)	2	1, 3	3	1

18. Match the following.

	Column-I	Column-II	
(P)	Let $f(x) = [x-1] + [1-x]$, $[x]$ is the greatest integer function, a is an integer, then	1.	continuous at $x = a$
(Q)	Let f be as in (P) but a is not an integer, then	2.	$\lim\limits_{x\to0} f(x)$ does not exist
(R)	Let $f(x) = \cot x$, then	3.	$f(a) = 0$
(S)	If $f(x)=\dfrac{a^{\cot x}-a^{\cos x}}{\cot x-\cos x}$, then	4.	$\lim\limits_{x\to\pi/2} f(x)=\log a$

Codes :

	P	Q	R	S		P	Q	R	S
(a)	4	1	2	3	(b)	2	1	4	3
(c)	1	3	2	4	(d)	3	1	2	4

19. Match the following.

	Column-I	Column-II	
(P)	The value of the expression $f(x)=\sin^2 x+\sin^2\left(x+\frac{\pi}{3}\right)+\cos x\cdot\cos\left(x+\frac{\pi}{3}\right)$, $\forall\ x\in R$ is equal to	1.	1/2
(Q)	Exact value of $\cos40°(1 - 2\sin10°)$ is	2.	$-1/3$
(R)	The value of $\sum\limits_{k=3}^{\infty}\sin^k\left(\frac{\pi}{6}\right)$ is	3.	1/4
(S)	The value of λ for which the lines are concurrent $x+y+1=0$; $3x+2\lambda y+4=0$; $x+y-3\lambda=0$ is/are	4.	5/4

Codes :

	P	Q	R	S		P	Q	R	S
(a)	4	3	1	2	(b)	3	4	1	2
(c)	1	3	2	4	(d)	4	1	3	2

20. Match the following.

	Column-I	Column-II	
(P)	Let a function f is defined as $f:\{1, 2, 3, 4\}\to\{1, 2, 3, 4\}$. If f satisfy $f(f(x)) = f(x)$, $\forall\ x\in\{1, 2, 3, 4\}$ then number of such functions is	1.	9
(Q)	If m and M are the least and greatest value of $f(x) = (\cos^{-1}x)^2 + (\sin^{-1}x)^2$, then M/m has the value equal to	2.	10
(R)	Let x and y be two real numbers such that $2\sin x\sin y + 3\cos y + 6\cos x\sin y = 7$. The value of $\tan^2x + 2\tan^2y$ is	3.	41

Codes :

	P	Q	R		P	Q	R
(a)	3	2	1	(b)	2	1	3
(c)	2	3	1	(d)	3	1	2

ANSWER KEYS

Paper-I

1. (a,b)	2. (b,d)	3. (b,c)	4. (a,b,c)	5. (a,d)
6. (a,b,c,d)	7. (c)	8. (a,b,c)	9. (b)	10. (a,d)
11. (1)	12. (4)	13. (1)	14. (1)	15. (2)
16. (6)	17. (3)	18. (2)	19. (5)	20. (3)

Paper-II

1. (c)	2. (d)	3. (a)	4. (c)	5. (c)
6. (b)	7. (a)	8. (d)	9. (a)	10. (d)
11. (b)	12. (c)	13. (d)	14. (d)	15. (b)
16. (c)	17. (d)	18. (d)	19. (d)	20. (a)

◈◈

MATHS MUSING

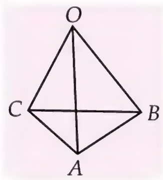
SOLUTION SET-180

1. **(a):** The sum of all two-digit numbers $= 10 + 11 + 12 + ... + 99 = \frac{90}{2}(10+99) = 4905$. There are $5 \times 5 = 25$ numbers with two odd digits. The sum of these numbers $= 5(1 + 3 + 5 + 7 + 9) \times 11 = 55 \times 25 = 1375$ Likewise, the sum of numbers with two even digits $= 5(0+2+4+6+8) \times 11 - (0+2+4+6+8) = 54 \times 20 = 1080$

$\therefore \; S = 4905 - 1375 - 1080 = 2450 = 2 \cdot 5^2 \cdot 7^2$

2. **(a):** $\overrightarrow{OA} = \vec{a}, \overrightarrow{OB} = \vec{b}, \overrightarrow{OC} = \vec{c}$, $|\vec{a}| = 3, |\vec{b} - \vec{c}| = 2$

The shortest distance between OA and BC is 2.



$\therefore \; |(\vec{b} - \vec{a}) \cdot \vec{a} \times (\vec{b} - \vec{c})| = 2|\vec{a} \times (\vec{b} - \vec{c})|$

$\Rightarrow \; [\vec{a}\,\vec{b}\,\vec{c}] = 2 \cdot 3 \cdot 2 \cdot \sin 30°$

$\therefore \; \text{Volume} = \frac{1}{6}[\vec{a}\,\vec{b}\,\vec{c}] = 1$

3. **(d):** $f(x) = 2^{(\log_8 3)\cos^2 x} + 3^{(\log_8 2)\sin^2 x}$

$= 2^{(\log_8 3)\cos^2 x} + 2^{(\log_8 3)\sin^2 x}$

$\Rightarrow f(x) \ge 2^1 \sqrt{(2^{(\log_8 3)\cos^2 x} \cdot 2^{(\log_8 3)\sin^2 x})}$ (using A.M. ≥ G.M.)

$= 2\left(2^{\log_8 3}\right)^{1/2} = 2^{1+\log_8 \sqrt{3}}$

4. **(a):** $z_1 = f\,|\,z_0\,| = \frac{z_0 + i}{z_0 - i}$, $z_2 = i\left(\frac{z_0 + 1}{z_0 - 1}\right), z_3 = z_0$

$\therefore \; z_{2011} = z_1 = \frac{z_0 + i}{z_0 - i} = \frac{\frac{1}{5} + i + i}{\frac{1}{5} + i - i} = 1 + 10i$

5. **(d):** Let $\angle ADC = \theta$. By cosine rule,

$AB^2 = m^2 = n^2 + 9^2 + 2 \cdot 9 \cdot n \cos\theta$

$AC^2 = m^2 = n^2 + (21)^2 - 2 \cdot 21 \cdot n \cos\theta$

Eliminating $\cos\theta$, we get $m^2 - n^2 = 189$

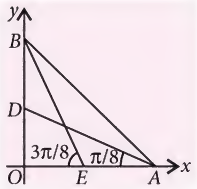
$\therefore \; m - n = 1, m + n = 189, m = 95$

$m - n = 3, m + n = 63, m = 33$

$m - n = 7, m + n = 27, m = 17$

6. **(c):** The angles are invariant under translation. We can take the vertices of the triangle as $O(0, 0)$, $A(1, 0)$, $B(0, 1)$.

AD and BE are the bisectors of $\angle A$ and $\angle B$.



$\angle OAD = \frac{\pi}{8}, \angle OEB = \frac{3\pi}{8}$

$m_1 = -\tan\frac{\pi}{8} = -(\sqrt{2} - 1)$

$m_2 = -\tan\frac{3\pi}{8} = -\left(\sqrt{2} + 1\right)$

$\therefore \; \frac{m_1}{m_2} = \frac{\sqrt{2} - 1}{\sqrt{2} + 1} = \left(\sqrt{2} - 1\right)^2 = 3 - 2\sqrt{2}$

7. **(b):** Let r be the radius of the circle and θ be the angle subtended by the chord AF at the centre.

$4 = 2r\sin\frac{\theta}{2}, \; 11 = 2r\sin\frac{5\theta}{2}$

$\therefore \; \frac{11}{4} = \frac{\sin\frac{5\theta}{2}}{\sin\frac{\theta}{2}} = \frac{\sin 3\theta + \sin 2\theta}{\sin\theta} = 3 - 4\sin^2\theta + 2\cos\theta$

$\Rightarrow \; 4\cos^2\theta + 2\cos\theta = \frac{15}{4}$

$\therefore \; \cos\theta = \frac{3}{4}, \sin\theta = \frac{\sqrt{7}}{4}, \sin\frac{\theta}{2} = \frac{1}{2\sqrt{2}}$

$4 = 2r \cdot \frac{1}{2\sqrt{2}} \; \Rightarrow \; r = 4\sqrt{2}$

$AE = 2r\sin\theta = 8\sqrt{2} \cdot \frac{\sqrt{7}}{4} = 2\sqrt{14}$

8. **(a):** $AD = 2r\sin\frac{3\theta}{2} = 10$

9. **(5):** $(1 + x)^5 = C_0 + C_1x + C_2x^2 + C_3x^3 + C_4x^4 + C_5x^5$

Differentiating

$5(1 + x)^4 = C_1 + 2C_2x + 3C_3x^2 + 4C_4x^3 + 5C_5x^4$

$\therefore \; 5\left(1 + \frac{1}{x}\right)^4 = C_1 + 2\frac{C_2}{x} + \frac{3C_3}{x^2} + \frac{4C_4}{x^3} + \frac{5C_5}{x^4}$

Multiplying the above two series and considering the coefficient of x is $\sum_{r=2}^{5}(r-1)r \cdot C_{r-1}C_r$ = coefficient of x in

$25\,\frac{(1+x)^8}{x^4} = 25\binom{8}{5} = 25\binom{8}{3} = 1400$

10. **(a):** **(P)** → **(1)**, **(Q)** → **(2)**, **(R)** → **(4)**, **(S)** → **(3)**

(P) We have, $12^x + 12^{x+1} = 3^x + 3^{x+1} + 3^{x+2}$

$\Rightarrow \; 12^x(13) = 3^x(13) \Rightarrow 3^x(4^x - 1) = 0$

$\Rightarrow \; 3^x = 0$ (not possible)

So, $4^x - 1 = 0 \; \Rightarrow \; 4^x = 1 \Rightarrow x = 0$

(Q) $\int_{\sqrt{2}}^{x} \frac{dt}{t\sqrt{t^2 - 1}} = \frac{\pi}{12} \Rightarrow \sec^{-1} x - \frac{\pi}{4} = \frac{\pi}{12}$ or $\sec^{-1} x = \frac{\pi}{3}, x = 2$

(R) $5(2\cos^2\theta - 1) + 1 + \cos\theta + 1 = 0$

$\Rightarrow \; 10\cos^2\theta + \cos\theta - 3 = 0 \; \Rightarrow \; \cos\theta = \frac{1}{2}, \; -\frac{3}{5}$

One values of θ in $(0, \pi/2)$.

(S) $3^{2n+2} - 8n - 9 = (1 + 8)^{n+1} - 8n - 9$

$= \binom{n+1}{2}8^2 + ...,$

which is divisible by $8^2 = 2^6 \; \Rightarrow \; m = 6$.